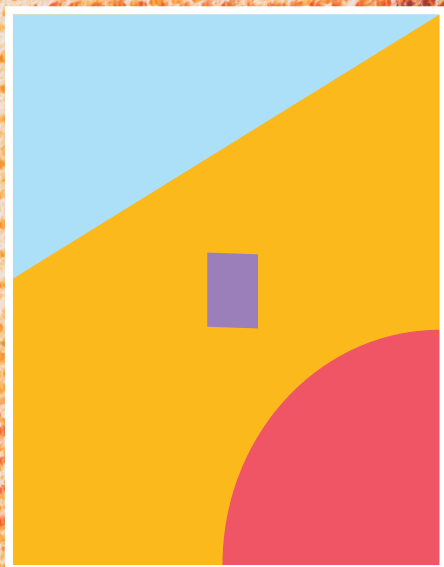


# Models in Microeconomic Theory

Expanded Second Edition

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# 5 Consumer behavior

In this chapter we apply the model of individual choice presented in [Chapter 2](#) to the behavior of a consumer. The set  $X$  of all alternatives the consumer may face is  $\mathbb{R}_+^2$ , the set of **bundles**, and a **choice problem** is a subset of  $X$ . As we discussed in [Chapter 2](#), to completely describe an individual's behavior we need to specify her choice for every **choice problem** she may face. Not every subset of  $X$  is a choice problem for a consumer. Since we are interested in the connection between prices and the consumer's choices, we focus on the behavior of a consumer who faces a particular type of choice problem, called a budget set.

## 5.1 Budget sets

A choice problem for a consumer is the set of bundles that she can purchase, given the prices and her wealth. We refer to this set as the consumer's budget set. More precisely, given prices  $p_1$  and  $p_2$  and wealth  $w$ , the consumer's budget set is the set of all bundles the consumer can obtain by exchanging  $w$  for the goods at the fixed exchange rates of  $p_1$  units of wealth for one unit of good 1 and  $p_2$  units of wealth for one unit of good 2.

### Definition 5.1: Budget set

For any positive numbers  $p_1$ ,  $p_2$ , and  $w$ , the *budget set* of a consumer with wealth  $w$  when the prices are  $(p_1, p_2)$  is

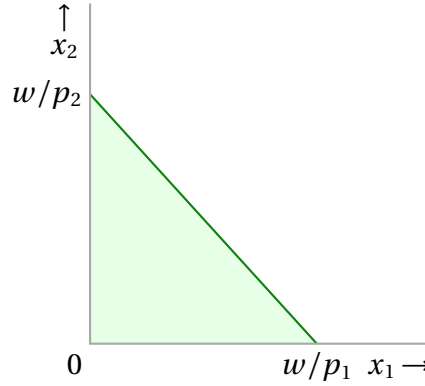
$$B((p_1, p_2), w) = \{(x_1, x_2) \in X : p_1 x_1 + p_2 x_2 \leq w\}.$$

The set  $\{(x_1, x_2) \in X : p_1 x_1 + p_2 x_2 = w\}$  is the consumer's *budget line*.

Geometrically, a budget set is a triangle like the one in [Figure 5.1](#). Note that multiplying wealth and prices by the same positive number does not change the set:  $B((\lambda p_1, \lambda p_2), \lambda w) = B((p_1, p_2), w)$  for any  $\lambda > 0$ , because the inequalities  $\lambda p_1 x_1 + \lambda p_2 x_2 \leq \lambda w$  and  $p_1 x_1 + p_2 x_2 \leq w$  that define these sets are equivalent.

Every budget set is convex: if  $a$  and  $b$  are in  $B((p_1, p_2), w)$  then  $p_1 a_1 + p_2 a_2 \leq w$  and  $p_1 b_1 + p_2 b_2 \leq w$ , so that for any  $\lambda \in [0, 1]$  we have

$$\begin{aligned} p_1(\lambda a_1 + (1 - \lambda)b_1) + p_2(\lambda a_2 + (1 - \lambda)b_2) \\ = \lambda(p_1 a_1 + p_2 a_2) + (1 - \lambda)(p_1 b_1 + p_2 b_2) \leq w, \end{aligned}$$



**Figure 5.1** The light green triangle is the budget set of a consumer with wealth  $w$  when the prices of the goods are  $p_1$  and  $p_2$ ,  $\{(x_1, x_2) \in X : p_1x_1 + p_2x_2 \leq w\}$ . The dark green line is the budget line.

and hence  $(\lambda a_1 + (1 - \lambda)b_1, \lambda a_2 + (1 - \lambda)b_2)$  is in  $B((p_1, p_2), w)$ .

Figure 5.1 shows also the budget line: the set of all bundles  $(x_1, x_2)$  satisfying  $p_1x_1 + p_2x_2 = w$ , a line with negative slope. The equation of this line can be alternatively expressed as  $x_2 = (-p_1/p_2)x_1 + w/p_1$ . The slope of the line,  $-p_1/p_2$ , expresses the tradeoff the consumer faces: consuming one more unit of good 1 requires consuming  $p_1/p_2$  fewer units of good 2.

Thus every **choice problem** of the consumer is a right triangle with two sides on the axes. Every such triangle is generated by some pair  $((p_1, p_2), w)$ . The same collection of choice problems is generated also in a different model of the consumer's environment. Rather than assuming that the consumer can purchase the goods at given prices using her wealth, assume that she initially owns a bundle  $e$  and can exchange goods at the fixed rate of one unit of good 1 for  $\beta$  units of good 2. Then her choice problem is  $\{(x_1, x_2) \in X : (e_1 - x_1)\beta \geq x_2 - e_2\}$  or  $\{(x_1, x_2) \in X : \beta x_1 + x_2 \leq \beta e_1 + e_2\}$ , which is equal to the budget set  $B((\beta, 1), \beta e_1 + e_2)$ .

## 5.2 Demand functions

A consumer's **choice function**, called a demand function, assigns to every budget set one of its members. A budget set here is defined by a pair  $((p_1, p_2), w)$  with  $p_1$ ,  $p_2$ , and  $w$  positive. (In some later chapters it is specified differently.) Thus, a consumer's behavior can be described as a function of  $((p_1, p_2), w)$ .

### Definition 5.2: Demand function

A *demand function* is a function  $x$  that assigns to each **budget set** one of its members. Define  $x((p_1, p_2), w)$  to be the bundle assigned to the budget set  $B((p_1, p_2), w)$ .

Note that  $x((\lambda p_1, \lambda p_2), \lambda w) = x((p_1, p_2), w)$  for all  $((p_1, p_2), w)$  and all  $\lambda > 0$  because  $B((\lambda p_1, \lambda p_2), \lambda w) = B((p_1, p_2), w)$ .

The definition does not assume that the demand function is the result of the consumer's maximizing a preference relation. We are interested also in patterns of behavior that are not derived from such optimization.

Here are some examples of demand functions that reflect simple rules of behavior. In each case, the function  $x$  satisfies  $x((\lambda p_1, \lambda p_2), \lambda w) = x((p_1, p_2), w)$  for all  $((p_1, p_2), w)$  and all  $\lambda > 0$ , as required.

#### Example 5.1: All wealth spent on the cheaper good

The consumer purchases only the cheaper good; if the prices of the goods are the same, she divides her wealth equally between the two. Formally,

$$x((p_1, p_2), w) = \begin{cases} (w/p_1, 0) & \text{if } p_1 < p_2 \\ (w/(2p_1), w/(2p_2)) & \text{if } p_1 = p_2 \\ (0, w/p_2) & \text{if } p_1 > p_2. \end{cases}$$

#### Example 5.2: Equal amounts of the goods

The consumer chooses the same quantity of each good and spends all her wealth, so that  $x((p_1, p_2), w) = (w/(p_1 + p_2), w/(p_1 + p_2))$ .

#### Example 5.3: Half of wealth spent on each good

The consumer spends half of her wealth on each good, so that  $x((p_1, p_2), w) = (w/(2p_1), w/(2p_2))$ .

#### Example 5.4: Purchase one good up to a limit

The consumer buys as much as she can of good 1 up to 7 units and with any wealth remaining buys good 2. That is,

$$x((p_1, p_2), w) = \begin{cases} (w/p_1, 0) & \text{if } w/p_1 \leq 7 \\ (7, (w - 7p_1)/p_2) & \text{otherwise.} \end{cases}$$

### 5.3 Rational consumer

A rational consumer has a fixed [preference relation](#), and for any [budget set](#) chooses the best bundle in the set according to the preference relation. We refer

to the problem of finding the best bundle in a budget set according to a given preference relation as the consumer's problem.

### Definition 5.3: Consumer's problem

For a **preference relation**  $\succsim$  on  $\mathbb{R}_+^2$  and positive numbers  $p_1$ ,  $p_2$ , and  $w$ , the *consumer's problem* is the problem of finding the best bundle in the **budget set**  $B((p_1, p_2), w)$  according to  $\succsim$ . If  $\succsim$  is **represented by the utility function**  $u$ , this problem is

$$\max_{(x_1, x_2) \in X} u(x_1, x_2) \text{ subject to } p_1 x_1 + p_2 x_2 \leq w.$$

The following result gives some basic properties of a consumer's problem.

### Proposition 5.1: Solution of consumer's problem

Fix a **preference relation** on  $\mathbb{R}_+^2$  and a **budget set**.

- If the preference relation is **continuous** then the **consumer's problem** has a solution.
- If the preference relation is **strictly convex** then the **consumer's problem** has at most one solution.
- If the preference relation is **monotone** then any solution of the **consumer's problem** is on the **budget line**.

### Proof

- If the preference relation is continuous then it has a continuous utility representation (see Comment 1 on page 50). Given that both prices are positive, the budget set is compact, so that by a standard mathematical result the continuous utility function has a maximizer in the budget set, which is a solution of the consumer's problem.
- Assume that distinct bundles  $a$  and  $b$  are both solutions to a consumer's problem. Then the bundle  $(a+b)/2$  is in the budget set (which is convex); by the strict convexity of the preference relation this bundle is strictly preferred to both  $a$  and  $b$ .
- Suppose that the bundle  $(a_1, a_2)$  is in the consumer's budget set for prices  $p_1$  and  $p_2$  and wealth  $w$ , but is not on the budget line. Then

$p_1 a_1 + p_2 a_2 < w$ , so there exists  $\varepsilon > 0$  small enough that  $p_1(a_1 + \varepsilon) + p_2(a_2 + \varepsilon) < w$ , so that  $(a_1 + \varepsilon, a_2 + \varepsilon)$  is in the budget set. By the monotonicity of the preference relation,  $(a_1 + \varepsilon, a_2 + \varepsilon)$  is preferred to  $(a_1, a_2)$ , so that  $(a_1, a_2)$  is not a solution of the consumer's problem.

The next two examples give explicit solutions of the consumer's problem for specific preference relations.

#### Example 5.5: Complementary goods

Consider a consumer with a preference relation represented by the utility function  $\min\{x_1, x_2\}$  (see Example 4.4). This preference relation is **monotone**, so a solution  $(x_1, x_2)$  of the consumer's problem lies on the budget line:  $p_1 x_1 + p_2 x_2 = w$ . Since  $p_1 > 0$  and  $p_2 > 0$ , any solution  $(x_1, x_2)$  also has  $x_1 = x_2$ . If, for example,  $x_1 > x_2$ , then for  $\varepsilon > 0$  small enough the bundle  $(x_1 - \varepsilon, x_2 + \varepsilon p_1/p_2)$  is in the budget set and is preferred to  $x$ . Thus the consumer's problem has a unique solution  $(w/(p_1 + p_2), w/(p_1 + p_2))$ . Notice that the consumer's problem has a unique solution even though the preference relation is only **convex**, not strictly convex.

#### Example 5.6: Substitutable goods

A consumer wants to maximize the sum of the amounts of the two goods. That is, her preference relation is represented by  $x_1 + x_2$  (Example 4.1 with  $v_1 = v_2$ ). Such a preference relation makes sense if the two goods differ only in ways irrelevant to the consumer. When  $p_1 \neq p_2$ , a unique bundle solves the consumer's problem:  $(w/p_1, 0)$  if  $p_1 < p_2$  and  $(0, w/p_2)$  when  $p_1 > p_2$ . When  $p_1 = p_2$ , all bundles on the budget line are solutions of the consumer's problem.

### 5.4 Differentiable preferences

If a consumer's **preference relation** is **monotone**, **convex**, and **differentiable**, then for any bundle  $z$  the local valuations  $v_1(z)$  and  $v_2(z)$  represent the value of each good to the consumer at  $z$ . Thus a small change in the bundle  $z$  is an improvement for the consumer if and only if the change increases the value of the bundle measured by the local valuations at  $z$ . The consumer finds it desirable to give up a small amount  $\alpha$  of good 1 in return for an additional amount  $\beta$  of good 2 if and only if  $-\alpha v_1(z) + \beta v_2(z) > 0$ , or  $\beta/\alpha > v_1(z)/v_2(z)$ . Similarly, she finds it desirable to give up a small amount  $\beta$  of good 2 in return for an additional amount

$\alpha$  of good 1 if and only if  $\alpha v_1(z) - \beta v_2(z) > 0$ , or  $\beta/\alpha < v_1(z)/v_2(z)$ . The ratio  $v_1(z)/v_2(z)$  is called her marginal rate of substitution at  $z$ .

#### Definition 5.4: Marginal rate of substitution

For a **monotone**, **convex**, and **differentiable preference relation** on  $\mathbb{R}_+^2$  and **bundle**  $z$ , the *marginal rate of substitution* at  $z$ , denoted  $MRS(z)$ , is  $v_1(z)/v_2(z)$ , where  $v_1(z)$  and  $v_2(z)$  are the consumer's **local valuations** at  $z$ .

The following result characterizes the solution of the consumer's problem if the consumer's preference relation is monotone, convex, and differentiable.

#### Proposition 5.2: Marginal rate of substitution and price ratio

Assume that a consumer has a **monotone**, **convex**, and **differentiable preference relation** on  $\mathbb{R}_+^2$ . If  $x^*$  is a solution of the **consumer's problem** for  $(p_1, p_2, w)$  then

- a.  $x_1^* > 0$  and  $x_2^* > 0 \Rightarrow MRS(x^*) = p_1/p_2$
- b.  $x_1^* = 0 \Rightarrow MRS(x^*) \leq p_1/p_2$
- c.  $x_2^* = 0 \Rightarrow MRS(x^*) \geq p_1/p_2$ .

#### Proof

To show (a), denote the local valuations at  $x^*$  by  $v_1(x^*)$  and  $v_2(x^*)$ . Suppose that  $x_1^* > 0$ ,  $x_2^* > 0$ , and  $MRS(x^*) = v_1(x^*)/v_2(x^*) < p_1/p_2$ . For any  $\varepsilon > 0$  let  $y(\varepsilon) = x^* + (-\varepsilon, \varepsilon p_1/p_2)$ . (Refer to [Figure 5.2a](#).) Then

$$p_1 y_1(\varepsilon) + p_2 y_2(\varepsilon) = p_1(x_1^* - \varepsilon) + p_2(x_2^* + \varepsilon p_1/p_2) = w,$$

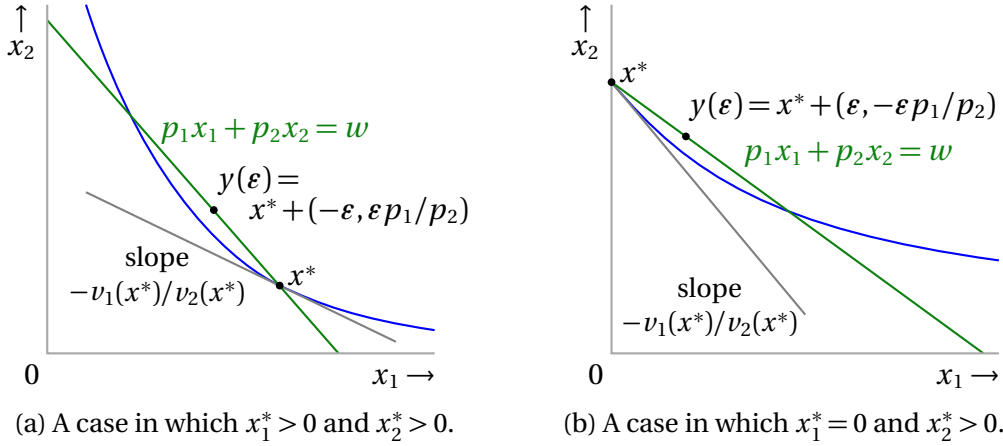
so that  $y(\varepsilon)$  is on the budget line. Also

$$v_1(x^*)(-\varepsilon) + v_2(x^*)(\varepsilon p_1/p_2) = -\varepsilon [v_1(x^*) - (p_1/p_2)v_2(x^*)] > 0.$$

Thus by the differentiability of the preference relation, there exists  $\varepsilon > 0$  such that  $y(\varepsilon) \succ x^*$ , contradicting the fact that  $x^*$  is a solution of the consumer's problem.

Similar arguments establish results (b) and (c) (refer to [Figure 5.2b](#)). Notice that if a bundle  $x^*$  with  $x_1^* = 0$  is optimal, then the inequality  $MRS(x^*) < p_1/p_2$  does not contradict the optimality of  $x^*$  because the consumer cannot reduce her consumption of good 1 in exchange for some amount of good 2. Therefore the optimality of  $x^*$  implies only the inequality  $MRS(x^*) \leq p_1/p_2$  and not the equality  $MRS(x^*) = p_1/p_2$ .





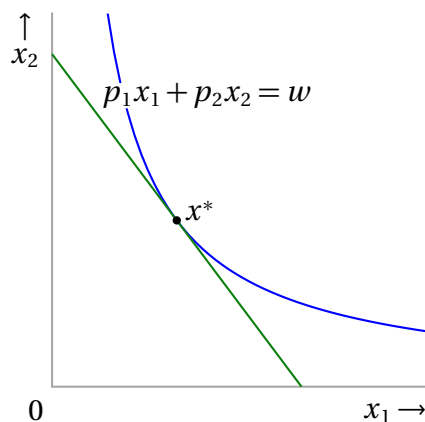
**Figure 5.2** An illustration of the proof of [Proposition 5.2](#).

### Example 5.7

A consumer's preference relation is represented by the utility function  $x_1 x_2$ . (An indifference curve is shown in [Figure 5.3](#).) Any bundle  $(x_1, x_2)$  with  $x_1 > 0$  and  $x_2 > 0$  is preferred to any bundle with  $x_1 = 0$  or  $x_2 = 0$ , so if  $x^*$  is a solution of the consumer's problem then  $x_1^* > 0$  and  $x_2^* > 0$ . The preference relation is [monotone](#), [convex](#), and [differentiable](#), so by [Proposition 5.2](#),  $MRS(x^*) = p_1/p_2$ . [Proposition 4.4](#) implies that  $MRS(x^*)$  is the ratio of the partial derivatives of the utility function at  $x^*$ , namely  $MRS(x^*) = x_2^*/x_1^*$ . Thus  $x_2^*/x_1^* = p_1/p_2$ , so that  $p_1 x_1^* = p_2 x_2^*$ . Since the preference relation is monotone, by [Proposition 5.1c](#)  $x^*$  lies on the budget line:  $p_1 x_1^* + p_2 x_2^* = w$ . Therefore  $(x_1^*, x_2^*) = (w/(2p_1), w/(2p_2))$ : the consumer spends half her wealth on each good.

### Example 5.8

A consumer's preference relation is represented by the utility function  $x_1 + \sqrt{x_2}$ . This preference relation is [monotone](#), [convex](#), and [differentiable](#), so that by [Proposition 5.1c](#) a solution of the consumer's problem is on the budget line and satisfies the conditions in [Proposition 5.2](#). We have  $MRS(x_1, x_2) = 2\sqrt{x_2}$ , so as  $x_1$  increases and  $x_2$  decreases along the budget line,  $MRS(x_1, x_2)$  decreases from  $2\sqrt{w/p_2}$  to 0. (See [Figure 5.4](#).) Hence if  $2\sqrt{w/p_2} \geq p_1/p_2$  the unique solution  $(x_1^*, x_2^*)$  of the consumer's problem satisfies  $MRS(x_1^*, x_2^*) = 2\sqrt{x_2^*} = p_1/p_2$ , or  $x_2^* = p_1^2/(4p_2^2)$  and  $x_1^* = w/p_1 - p_1/(4p_2)$ . If  $2\sqrt{w/p_2} \leq p_1/p_2$ , we have  $(x_1^*, x_2^*) = (0, w/p_2)$ : the consumer spends all her wealth on the second good.



**Figure 5.3** An indifference set for the preference relation in [Example 5.7](#).

## 5.5 Rationalizing a demand function

In the previous two sections, we study the [demand function](#) obtained from the maximization of a preference relation. We now study whether a demand function, which could be the outcome of a different procedure, is consistent with the consumer's maximization of a preference relation. That is, rather than deriving a demand function from a preference relation, we go in the opposite direction: for a given demand function, we ask whether there exists a [monotone](#) preference relation such that the solutions of the [consumer's problem](#) are consistent with the demand function. Or, more compactly, we ask which demand functions are rationalized by preference relations.

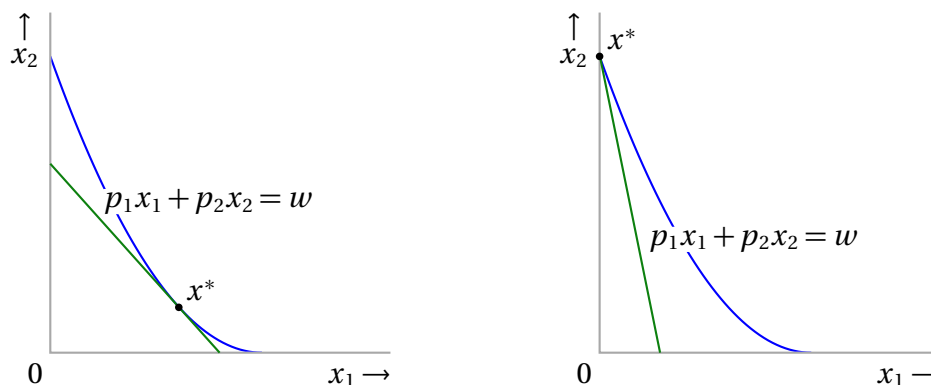
### Definition 5.5: Rationalizable demand function

A [demand function](#) is *rationalizable* if there is a [monotone preference relation](#) such that for every [budget set](#) the alternative specified by the demand function is a solution of the [consumer's problem](#).

Note that the definition does not require that the alternative specified by the demand function is the *only* solution of the consumer's problem.

In [Section 2.2](#) we show (Propositions [2.1](#) and [2.2](#)) that a choice function is [rationalizable](#) if and only if it satisfies [property  \$\alpha\$](#) . These results have no implications for a consumer's demand function, because property  $\alpha$  is vacuous in this context: if the bundle  $a$  is chosen from budget set  $B$  and is on the frontier of  $B$  then no other budget set that contains  $a$  is a subset of  $B$ .

We now give some examples of demand functions and consider preference relations that rationalize them.



- (a) A case in which the solution  $x^*$  has  $x_1^* > 0$  and  $MRS(x_1^*, x_2^*) = p_1/p_2$ .  
 (b) A case in which the solution  $x^*$  has  $x_1^* = 0$  and  $MRS(x_1^*, x_2^*) > p_1/p_2$ .

**Figure 5.4** Solutions of the consumer's problem in [Example 5.8](#).

#### Example 5.9: All wealth spent on the cheaper good

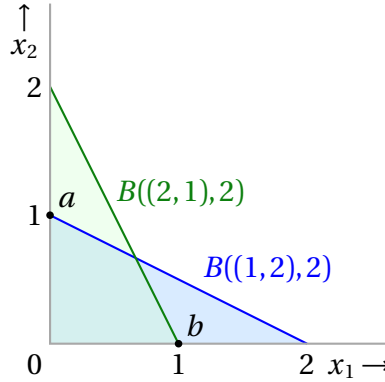
Consider the demand function in [Example 5.1](#). That is, if the prices of the good differ, the consumer spends all her wealth on the cheaper good; if the prices are the same, she splits her wealth equally between the two goods. This demand function is rationalized by the preference relation represented by  $x_1 + x_2$ . It is also rationalized by the preference relations represented by  $\max\{x_1, x_2\}$  and by  $x_1^2 + x_2^2$ .

#### Example 5.10: Half of wealth spent on each good

Consider the demand function in [Example 5.3](#). That is, the consumer spends half of her wealth on each good, independently of the prices and her wealth. This demand function is rationalized by the preference relation represented by the function  $x_1x_2$  ([Example 5.7](#)). Thus although maximizing the product of the quantities of the goods may seem odd, this function rationalizes a natural demand function.

#### Example 5.11: All wealth spent on the more expensive good

The consumer spends all her wealth on the more expensive good; if the prices are the same, she buys only good 1. That is,  $x((p_1, p_2), w) = (w/p_1, 0)$  if  $p_1 \geq p_2$  and  $(0, w/p_2)$  if  $p_1 < p_2$ . This demand function *cannot* be rationalized by a [monotone](#) preference relation. The consumer chooses the bundle  $a$  from  $B((1, 2), 2)$  and the bundle  $b$  from  $B((2, 1), 2)$  (see [Figure 5.5](#)).



**Figure 5.5** An illustration of the demand function in [Example 5.11](#). The light blue triangle is the budget set  $B((1, 2), 2)$  and the light green triangle is the budget set  $B((2, 1), 2)$ .

Since  $b$  is in the interior of  $B((1, 2), 2)$ , we have  $a \succ b$  for any monotone preference relation  $\succsim$  that rationalizes the demand function. Similarly,  $b \succ a$  because  $a$  is in the interior of  $B((2, 1), 2)$ , a contradiction.

The demand function in this last example might make sense in environments in which the price of a good reveals information about the quality of the good, or consumers like the prestige of consuming an expensive good. The example highlights a hidden assumption in the model of consumer behavior: prices do not convey information about the quality of the goods, and an individual's preferences are not affected by the prices and her wealth.

#### *The weak axiom of revealed preference*

If an individual chooses alternative  $a$  when alternative  $b$  is available, we might conclude that she finds  $a$  to be at least as good as  $b$ . If she chooses  $a$  when  $b$  is available and costs less than  $a$ , we might similarly conclude, if the goods are desirable, that she prefers  $a$  to  $b$ . (See [Figure 5.6a](#).) For example, if an individual purchases the bundle  $(2, 0)$  when she could have purchased the bundle  $(0, 2)$ , then we conclude that she finds  $(2, 0)$  at least as good as the bundle  $(0, 2)$  and prefers  $(2, 0)$  to  $(0, 1.9)$ .

#### **Definition 5.6: Revealed preference**

Given the [demand function](#)  $x$ , the bundle  $a$  is *revealed to be at least as good as the bundle*  $b$  if for some prices  $(p_1, p_2)$  and wealth  $w$  the [budget set](#)  $B((p_1, p_2), w)$  contains both  $a$  and  $b$ , and  $x((p_1, p_2), w) = a$ . The bundle  $a$  is *revealed to be better than*  $b$  if for some prices  $(p_1, p_2)$  and wealth  $w$



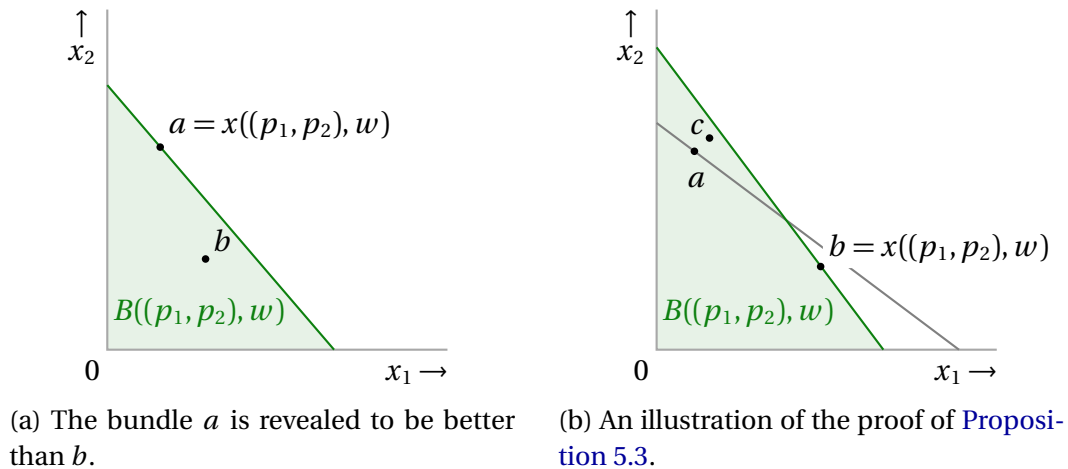


Figure 5.6

the budget set  $B((p_1, p_2), w)$  contains both  $a$  and  $b$ ,  $p_1 b_1 + p_2 b_2 < w$ , and  $x((p_1, p_2), w) = a$ .

We now define a property that is satisfied by every demand function rationalized by a monotone preference relation.

#### Definition 5.7: Weak axiom of revealed preference (WARP)

A demand function satisfies the *weak axiom of revealed preference* (WARP) if for no bundles  $a$  and  $b$ , both  $a$  is revealed to be at least as good as  $b$  and  $b$  is revealed to be better than  $a$ .

#### Proposition 5.3: Demand function of rational consumer satisfies WARP

A demand function that is rationalized by a monotone preference relation satisfies the weak axiom of revealed preference.

#### Proof

Let  $x$  be a demand function that is rationalized by the monotone preference relation  $\succsim$ . Assume, contrary to the result, that (i)  $a$  is revealed to be at least as good as  $b$  and (ii)  $b$  is revealed to be better than  $a$ . Given (i), we have  $a \succsim b$ . By (ii) there are prices  $p_1$  and  $p_2$  and wealth  $w$  such that  $b = x((p_1, p_2), w)$  and  $p_1 a_1 + p_2 a_2 < w$ . Let  $c$  be a bundle in  $B((p_1, p_2), w)$  with  $c_1 > a_1$  and  $c_2 > a_2$ . (Refer to Figure 5.6b.) By the monotonicity of the preference relation we have  $c \succ a$ , and since  $b$  is chosen from  $B((p_1, p_2), w)$

we have  $b \succ c$ . It follows from the **transitivity** of the preference relation that  $b \succ a$ , contradicting  $a \succ b$ .

Propositions 2.1 and 2.2 show that for a general choice problem, a choice function is rationalizable if and only if it satisfies property  $\alpha$ . Notice that by contrast, Proposition 5.3 provides only a necessary condition for a demand function to be rationalized by a monotone preference relation, not a sufficient condition. We do not discuss a sufficient condition, called the strong axiom of revealed preference.

## 5.6 Properties of demand functions

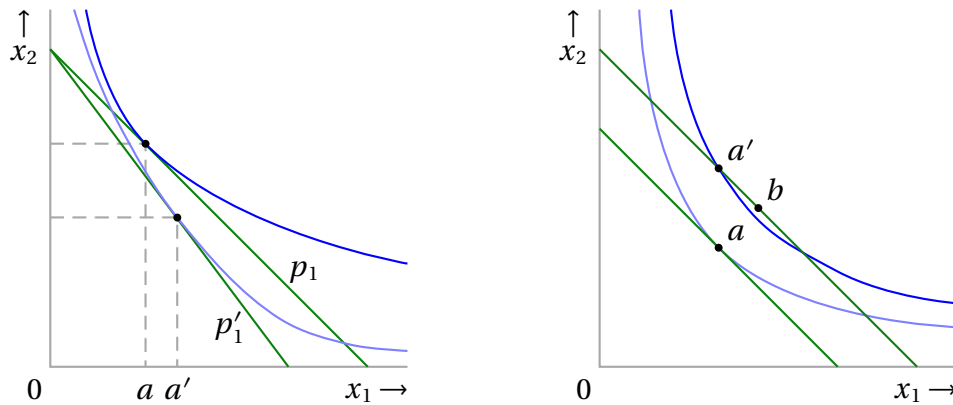
A demand function describes the bundle chosen by a consumer as a function of the prices and the consumer's wealth. If we fix the price of good 2 and the consumer's wealth, the demand function describes how the bundle chosen by the consumer varies with the price of good 1. This relation between the price of good 1 and its demand is called the consumer's regular, or Marshallian, demand function for good 1 (given the price of good 2 and wealth). The relation between the price of good 2 and the demand for good 1 (given a price of good 1 and the level of wealth), is called the consumer's cross-demand function for good 2. And the relation between the consumer's wealth and her demand for good  $i$  (given the prices) is called the consumer's Engel function for good  $i$ .

### Definition 5.8: Regular, cross-demand, and Engel functions

Let  $x$  be the **demand function** of a consumer.

- For any given price  $p_2^0$  of good 2 and wealth  $w^0$ , the function  $x_1^*$  defined by  $x_1^*(p_1) = x_1((p_1, p_2^0), w^0)$  is the consumer's *regular (or Marshallian) demand function for good 1 given  $(p_2^0, w^0)$* , and the function  $\hat{x}_2$  defined by  $\hat{x}_2(p_1) = x_2((p_1, p_2^0), w^0)$  is the consumer's *cross-demand function for good 2 given  $(p_2^0, w^0)$* .
- For any given prices  $(p_1^0, p_2^0)$ , the function  $\bar{x}_k$  defined by  $\bar{x}_k(w) = x_k((p_1^0, p_2^0), w)$  is the consumer's *Engel function for good  $k$  given  $(p_1^0, p_2^0)$* .

Consider, for example, a consumer who spends the fraction  $\alpha$  of her budget on good 1 and the rest on good 2, so that  $x((p_1, p_2), w) = (\alpha w/p_1, (1 - \alpha)w/p_2)$ . The consumer's regular demand function for good 1 given  $p_2^0$  and  $w^0$  is given by  $x_1^*(p_1) = \alpha w^0/p_1$  (and in particular does not depend on  $p_2^0$ ), her cross-demand function for good 2 given  $p_2^0$  and  $w^0$  is the constant function  $\hat{x}_2(p_1) =$



(a) An example in which the demand for good 1 increases when the price of good 1 increases.

(b) An illustration of the proof of Proposition 5.4.

Figure 5.7

$(1 - \alpha)w^0/p_2^0$ , and her Engel function for good 1 given the prices  $(p_1^0, p_2^0)$  is the linear function  $\bar{x}_1(w) = \alpha w/p_1^0$ .

We now introduce some terminology for various properties of the demand function.

**Definition 5.9: Normal, regular, and Giffen goods**

A good is *normal* for a given consumer if, for any given prices, the consumer's Engel function for the good is increasing. A good is *regular* for the consumer if for any price of the other good and any wealth, the consumer's regular demand function for the good is decreasing, and *Giffen* if her regular demand function for the good is increasing.

This terminology can also be applied locally: we say, for example, that good 1 is a Giffen good at  $((p_1^0, p_2^0), w^0)$  if the demand function is increasing around  $p_1^0$  given  $p_2^0$  and  $w^0$  (but is not necessarily increasing at all values of  $p_1$ ).

It is common to assume that every good is regular: as the price of the good increases, given the price of the other good and the consumer's wealth, the consumer's demand for the good decreases. The demand function of a rational consumer whose preference relation satisfies the standard assumptions of [monotonicity](#), [continuity](#), [convexity](#) and [differentiability](#) does not necessarily have this property. We do not give an explicit example but Figure 5.7a is suggestive: as the price of good 1 increases from  $p_1$  to  $p'_1$ , given the price of good 2 and the consumer's wealth, the consumer's demand for good 1 increases from  $a$  to  $a'$ .

The following result gives a condition on the preference relation that guarantees that a good is normal.

**Proposition 5.4: MRS and normal good**

The demand function of a rational consumer whose marginal rate of substitution  $MRS(x_1, x_2)$  is increasing in  $x_2$  for every value of  $x_1$  has the property that good 1 is normal (the consumer's Engel function for the good is increasing).

**Proof**

Fix  $p_1^0$  and  $p_2^0$  and let  $w' > w$ . Let  $a$  be a solution of the consumer's problem for the budget set  $B((p_1^0, p_2^0), w)$  and let  $a'$  be a bundle on the frontier of  $B((p_1^0, p_2^0), w')$  with  $a'_1 = a_1$ . (Refer to Figure 5.7b.) By the assumption on the marginal rate of substitution,  $MRS(a') > MRS(a)$ , and hence the solution,  $b$ , of the consumer's problem for the budget set  $B((p_1^0, p_2^0), w')$  has  $b_1 > a'_1 = a_1$ .

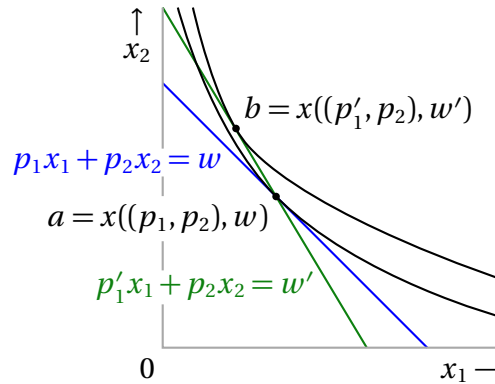
The analysis in this section compares the choices of a consumer for two sets of parameters. Such analyses are called comparative statics. The word statics refers to the fact that the comparison does not involve an analysis of the path taken by the outcome through time as the parameters change; we simply compare one outcome with the other. For example, the properties of the regular demand function can be viewed as answering the comparative statics question of how a consumer's behavior differs for two sets of parameters (prices and wealth) that differ only in the price of one of the goods. Phenomena related to the fact that people's behavior when they confront one budget set depends also on their behavior in a budget set they faced previously are not captured by this exercise.

The following comparative static result involves a consumer whose demand function satisfies the weak axiom of revealed preference and who chooses a bundle on the budget line. The result considers the effect of a change in the price of a good when the consumer's wealth is adjusted so that she has exactly enough wealth to purchase the bundle she chose before the change. The result asserts that if the consumer's wealth is adjusted in this way when the price of good 1 increases, then the consumer purchases less of good 1.

**Proposition 5.5: Slutsky property**

Assume that the demand function  $x$  of a rational consumer is single-valued, satisfies the weak axiom of revealed preference, and satisfies  $p_1 x_1((p_1, p_2), w) + p_2 x_2((p_1, p_2), w) = w$  for all  $((p_1, p_2), w)$ . Let  $p'_1 > p_1$  and let  $w'$  be the cost of the bundle  $x((p_1, p_2), w)$  for the pair of prices





**Figure 5.8** An illustration of the proof of [Proposition 5.5](#).

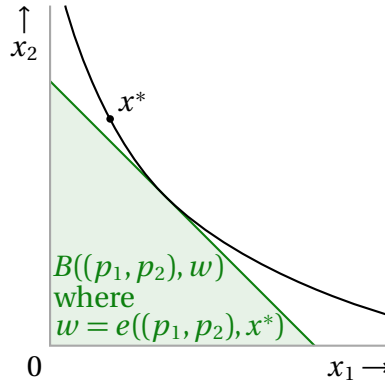
$(p'_1, p_2) : w' = p'_1 x_1((p_1, p_2), w) + p_2 x_2((p_1, p_2), w)$ . Then  $x_1((p'_1, p_2), w') \leq x_1((p_1, p_2), w)$ .

### Proof

Let  $a = x((p_1, p_2), w)$  and  $b = x((p'_1, p_2), w')$  (see [Figure 5.8](#)). By construction  $a$  is in the budget set  $B((p'_1, p_2), w')$  and therefore  $b$  is revealed to be at least as good as  $a$ . The slope of the budget line for the price  $p'_1$  is larger than the slope for the price  $p_1$ . Therefore if  $b_1 > a_1$  then  $b$  is in the interior of  $B((p_1, p_2), w)$ . As  $a$  is chosen from  $B((p_1, p_2), w)$  and  $b$  is interior,  $a$  is revealed to be better than  $b$ . This conclusion contradicts the assumption that the demand function satisfies the weak axiom of revealed preference.

### Problems

1. *Lexicographic preferences.* Find the demand function of a rational consumer with [lexicographic preferences](#) (with first priority on good 1).
2. *Cobb-Douglas preferences.* A consumer's preference relation is represented by the utility function  $x_1^\alpha x_2^{1-\alpha}$  where  $0 < \alpha < 1$ . These preferences are convex and differentiable. Show that for all prices and wealth levels the consumer spends the fraction  $\alpha$  of her budget on good 1.
3. *Rationalizing a demand function I.* Consider the demand function for which the consumer spends her entire wealth on the two goods and the ratio of the amount spent on good 1 to the amount spent on good 2 is  $p_2/p_1$ . Show that the preference relation represented by the utility function  $\sqrt{x_1} + \sqrt{x_2}$  rationalizes this demand function.



**Figure 5.9** The budget set  $B((p_1, p_2), e((p_1, p_2), x^*))$  (see Problem 5).

4. *Rationalizing a demand function II.* A consumer chooses the bundle at the intersection of the budget line and a ray from the origin orthogonal to the frontier. Can this demand function be rationalized by a monotone preference relation?
5. *Expenditure function.* A consumer's preference relation is monotone, continuous, and convex. Let  $x^* = (x_1^*, x_2^*)$  be a bundle. For any pair  $(p_1, p_2)$  of prices, let  $e((p_1, p_2), x^*)$  be the smallest wealth that allows the consumer to purchase a bundle that is at least as good for her as  $x^*$ :

$$e((p_1, p_2), x^*) = \min_{(x_1, x_2)} \{p_1 x_1 + p_2 x_2 : (x_1, x_2) \succeq (x_1^*, x_2^*)\}.$$

(See Figure 5.9.)

- a. Show that the function  $e$  is increasing in  $p_1$  (and  $p_2$ ).
- b. Show that for all  $\lambda > 0$  and every pair  $(p_1, p_2)$  of prices we have  $e((\lambda p_1, \lambda p_2), x^*) = \lambda e((p_1, p_2), x^*)$ .
6. *Rationalizing a demand function.* If the cost of buying 10 units of good 1 is less than  $\frac{1}{2}w$ , a consumer buys 10 units of good 1 and spends her remaining wealth on good 2. Otherwise she spends half of her wealth on each good. Show that this behavior is rationalized by a preference relation represented by the utility function

$$u(x_1, x_2) = \begin{cases} x_1 x_2 & \text{if } x_1 \leq 10 \\ 10 x_2 & \text{if } x_1 > 10. \end{cases}$$

7. *Consumer with additive utility function.* Suppose that the two goods are food ( $z$ ) and money ( $m$ ). A consumer's preference relation is represented by the

utility function  $m + v(z)$ , where  $v$  is increasing and concave and has a continuous derivative. The price of a unit of food in terms of money is  $\alpha$ . The consumer initially has  $M$  units of money and no food.

- a. Characterize the solution of the [consumer's problem](#).
- b. Compare the consumption of food of a consumer who faces two budget sets that differ only in the price of food. Show that when the price of food is  $\beta$  the amount of food consumed is not more than the amount consumed when the price of food is  $\alpha < \beta$ .

8. *Time preferences I.* Consider a consumer who chooses how much to consume at each of two dates and can transfer consumption from one date to the other by borrowing and lending. We can model her behavior by treating consumption at date 1 and consumption at date 2 as the two different goods in the model studied in this chapter.

Assume that the consumer is endowed with  $y$  units of money at each date and faces an interest rate  $r > 0$  for both borrowing and lending, so that she can exchange 1 unit at date 1 for  $1 + r$  units at date 2. The consumer's budget set is then

$$\{(x_1, x_2) \in X : (1 + r)x_1 + x_2 \leq (1 + r)y + y\}.$$

Denote the consumer's demand function by  $x(r, y)$ . Assume that (i)  $x$  satisfies the [weak axiom of revealed preference](#), (ii) for every pair  $(r, y)$  the consumer chooses a bundle on the budget line  $\{(x_1, x_2) \in X : (1 + r)x_1 + x_2 = (1 + r)y + y\}$ , and (iii) consumption at each date is a [normal good](#).

- a. Show that if the consumer borrows when the interest rate is  $r_1$  then she borrows less (and may even save) when the interest rate is  $r_2 > r_1$ .
- b. Show that if the consumer chooses to lend when the interest rate is  $r_1$  then she does not borrow when the interest rate is  $r_2 > r_1$ .

9. *Time preferences II.* For the same model as in the previous question, assume that the consumer has a preference relation  $\succsim$  that is [monotone](#), [continuous](#), [convex](#), and [differentiable](#).

- a. Say that the preference relation is *time neutral* if for all  $s$  and  $t$  we have  $(s, t) \sim (t, s)$ . (That is, for any amounts  $s$  and  $t$  of consumption, the consumer is indifferent between consuming  $s$  units at date 1 and  $t$  units at date 2 and consuming  $t$  units at date 1 and  $s$  at date 2.) Show that for all values of  $t$  we have  $MRS(t, t) = 1$ .

- b.* We say that the preference relation has *present bias* if whenever  $t > s$  we have  $(t, s) \succ (s, t)$ . Show that for all values of  $t$  we have  $MRS(t, t) \geq 1$ .
- c.* Show, using only the assumption that the preferences are present biased, that if  $r \leq 0$  then any solution of the **consumer's problem** has at least as much consumption at date 1 as it does at date 2.

## Notes

Giffen goods were named after Robert Giffen by [Marshall \(1895, 208\)](#). Engel functions are named after Ernst Engel. The Slutsky property is due to [Slutsky \(1915\)](#). The theory of revealed preference is due to [Samuelson \(1938\)](#). The exposition of the chapter draws upon [Rubinstein \(2006a, Lecture 5\)](#).