# Models in Microeconomic Theory

**Expanded Second Edition** 

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This book is available in two versions, one that uses feminine pronouns and one that uses masculine pronouns. This version uses feminine pronouns.

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# 🔵 A market

As in the previous chapter, a society consists of a set of individuals and a set of houses; each house can accommodate only one person and each person can occupy only one house. Different individuals may have different preferences over the houses, but everyone prefers to occupy any house than to be homeless.

In this chapter, unlike in the previous one, we assume that the ownership of a house is recognized and protected. Each house is initially owned by some individual. Houses can be exchanged only with the mutual consent of both owners; no individual can force another individual to give up her house.

The model allows us to introduce the central economic idea of prices as a means of guiding the individuals to a reallocation of the houses in which no group of individuals want to voluntarily exchange their houses.

# 9.1 Model

A society is defined as in the previous chapter. In particular, we assume that each individual cares only about the house that she occupies, not about the houses occupied by other individuals. Also, we continue to assume, for simplicity, that preferences are strict: no individual is indifferent between any two houses.

We study a model called a market, which differs from a jungle in that ownership replaces power and an initial pattern of ownership replaces the power relation. A market consists of a society and an allocation e, where e(i) is the house initially owned by individual i.

#### **Definition 9.1: Market**

A market  $\langle N, H, (\geq^i)_{i \in N}, e \rangle$  consists of a society  $\langle N, H, (\geq^i)_{i \in N} \rangle$  and an allocation *e* for the society, called the *initial allocation*, which represents the initial ownership of the houses.

# Example 9.1

Consider the market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$  in which  $N = \{1, 2, 3, 4\}$ ,  $H = \{A, B, C, D\}$ , and the individuals' preferences and the initial allocation are given in the following table. Each column indicates the preference ordering of

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an individual, with the individual's favorite house at the top; the initial allocation is highlighted.

Individuals				
1	2	3	4	
В	Α	Α	Α	
С	B	C	D	
D	С	D	В	
A	D	В	C	

Individual 1 initially owns house *A*, which is the one she least prefers and everyone else most prefers. Thus every other individual wants to exchange houses with individual 1. Individual 2 can offer her the most attractive exchange, because she initially owns house *B*, which is individual 1's favorite. Thus we might expect that the outcome includes an exchange between individuals 1 and 2.

#### Example 9.2

Consider the market in the following table.

Individuals				
1	2	3	4	
D	Α	Α	В	
С	D	C	С	
A	С	D	D	
В	B	В	Α	

Individuals 1 and 3 are interested in exchanging their houses, and so are individuals 2 and 4. These two exchanges lead to the allocation indicated in red in the left-hand table below. After the exchanges, no further reallocation within any group is mutually desirable.

Individuals		Individuals						
1	2	3	4		1	2	3	4
D	Α	Α	В		D	Α	Α	B
С	D	C	С		С	D	$\boldsymbol{C}$	С
$\boldsymbol{A}$	C	D	D		A	C	D	D
В	B	В	Α		В	B	В	Α

Another possible outcome of exchange is indicated in the right-hand table. This allocation may be achieved by an agreement between individuals 1, 2, and 4. Alternatively, it may be achieved by individual 1 first

exchanging her house, *A*, with individual 2, which leads individual 1 to hold *B*, and then exchanging *B* with individual 4. After the first exchange, individual 1 holds a house, *B*, that she does not like; but she knows that she can subsequently exchange it with individual 4 for *D*, her favorite house.

# Equilibrium of market

The central concept in this chapter is that of an equilibrium of a market. In an equilibrium, a number is attached to each house. We may interpret the number as the value or price of the house. Each individual can exchange the house she owns initially only for houses with lower or equal prices. An equilibrium satisfies two conditions. First, each individual chooses the house that is best for her among the houses with prices at most equal to the price of the house she initially owns. Second, the outcome is harmonious in the sense that the individuals' independent choices generate an allocation, in which each house is chosen by precisely one individual.

**Definition 9.2: Equilibrium of market** 

An *equilibrium* of the market  $\langle N, H, (\succeq^i)_{i \in \mathbb{N}}, e \rangle$  is a pair (p, a) where

- *p*, a *price system*, is a function that attaches a number *p*(*h*) (a *price*) to each house *h* ∈ *H*
- *a* is an assignment

such that

# optimality of choices

for every individual  $i \in N$ , the house a(i) maximizes *i*'s preference relation  $\succeq^i$  over her *budget set*  $\{h \in H : p(h) \le p(e(i))\}$ :

 $a(i) \succeq^i h$  for all  $h \in H$  with  $p(h) \le p(e(i))$ 

#### feasibility

*a* is an allocation.

Notice the structure of the definition, which is common to many definitions of equilibrium. First we specify the nature of a candidate for equilibrium, which in this case is a pair consisting of a price system and an assignment. Then we specify the conditions for such a candidate to be an equilibrium.

#### Example 9.3

Consider Example 9.2. The allocation that results from the first pair of exchanges, a = (C, D, A, B), is not an outcome of any market equilibrium, by the following argument. If there is a price system p such that (p, a) is an equilibrium then we need p(A) = p(C): since C must be in the budget set of individual 1, we need  $p(C) \le p(A)$ , and since A must be in the budget set of individual 3, we need  $p(A) \le p(C)$ . Similarly, p(B) = p(D). But if  $p(A) \ge p(D)$  then individual 1 chooses house D, which is her favorite, not C, and if p(A) < p(D) = p(B) then individual 2 chooses house A, not D. Thus for no price system p is (p, a) a market equilibrium.

The allocation that results from the second group of exchanges, b = (D, A, C, B), is the outcome of a market equilibrium with a price system p satisfying p(A) = p(B) = p(D) > p(C). (In fact, Proposition 9.5 implies that b is the only equilibrium allocation of this market.)

# Example 9.4: Market with common preferences

Consider a market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$  in which all individuals have the same preference relation:  $\succeq^i = \succeq$  for all  $i \in N$ . Let p be a price system that reflects  $\succeq$  in the sense that for any houses h and h', p(h) > p(h') if and only if  $h \succ h'$ . (In the terminology of Chapter 1, p is a utility function that represents  $\succeq$ .) Then the pair (p, e) is an equilibrium: the assignment eis an allocation; the budget set of each individual i consists of the house e(i) and all houses that are inferior according to the common preferences  $\succcurlyeq$ , and thus her most preferred house in this set is e(i). Notice that any equilibrium allocation a satisfies  $a(i) \succeq e(i)$  for all i and thus a = e.

If every individual has a different favorite house, then there is an equilibrium that assigns the same price to every house.

# Example 9.5: Market in which individuals have different favorite houses

Consider a market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$  in which each individual has a different favorite house. Then (p, a) is an equilibrium if p assigns the same price to all houses and for every individual i, a(i) is i's favorite house. For this price system all budget sets are equal to H, so that each individual optimally chooses her favorite house; since no two individuals have the same favorite house, a is an allocation. In fact, in any equilibrium allocation a each individual gets her favorite house. Otherwise, let  $h^*$  be a most expensive house in  $\{h \in H :$  *h* is not the favorite house of the individual *i* for whom a(i) = h}. Let *i*<sup>\*</sup> be the individual for whom  $a(i^*) = h^*$ . Then *i*<sup>\*</sup>'s favorite house is not more expensive than  $h^*$  and thus given that she can afford  $a(i^*)$  she can afford her favorite house, so that  $a(i^*)$  is not optimal for *i*<sup>\*</sup> in her budget set, a contradiction.

# Comments

- 1. Note that the notion of equilibrium does not require an individual to be aware of the preferences of the other individuals. Each individual has to know only the price system to make her choice.
- 2. The notion of equilibrium is static. If a society is at an equilibrium, there is no reason for it to move away. But we do not specify a process by which a society that is not at an equilibrium might move to an equilibrium.

Any allocation can be transformed into any other allocation by implementing a set of trading cycles, each of which is a rotation of houses within a set of individuals.

For example, the move from (A, B, C, D) to (C, D, A, B) can be achieved by individual 1 exchanging her house with individual 3 and individual 2 exchanging her house with individual 4. In this case, each trading cycle consists of a single bilateral exchange; we denote these cycles by (1,3) and (2,4).

As another example, the move from (A, B, C, D) to (D, A, C, B) can be achieved by individuals 1, 2, and 4 agreeing on a rotation of the houses they initially own so that individual 1 get 4's house, 4 gets 2's house, and 2 gets 1's house, while individual 3 keeps her house. We denote these trading cycles by (1,4,2) and (3). Note that the trading cycle (1,4,2) can be achieved also by individual 1 first exchanging her house with individual 4, to yield the allocation (D, B, C, A), and then individual 4 exchanging her house (which is now A) with individual 2.

In general, a trading cycle is a sequence  $(i_1, ..., i_k)$  of individuals, with the interpretation that (either by simultaneous rotations of houses or by a sequence of bilateral exchanges) individual  $i_j$  gets the house originally owned by  $i_{j+1}$  for j = 1, ..., k-1 and  $i_k$  gets the house owned initially by  $i_1$ . A trading cycle consisting of a single individual, for example  $(i_1)$ , means that the individual keeps the house she owns.

Definition 9.3: Trading cycle and trading partition

A *trading cycle* in a market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$  is a finite sequence of distinct individuals (members of *N*). A *trading partition* is a set of trading cycles such that every individual belongs to exactly one of the cycles. A

trading partition  $\{(i_1^1, ..., i_{k_1}^1), ..., (i_1^m, ..., i_{k_m}^m)\}$  transforms the allocation *a* to the allocation *b* for which for each j = 1, ..., m we have  $b(i_l^j) = a(i_{l+1}^j)$  for  $l = 1, ..., k_j - 1$  and  $b(i_{k_j}^j) = a(i_1^j)$ .

We now show that for any pair of allocations, a unique trading partition transforms one allocation to the other.

# Lemma 9.1: Uniqueness of transforming trading partition

For any allocations *a* and *b* in a market, a unique trading partition transforms *a* to *b*.

#### Proof

We construct the trading partition *T* inductively. Start with an arbitrary individual  $i_1$ . If  $b(i_1) = a(i_1)$ , add the (degenerate) trading cycle  $(i_1)$  to *T*. Otherwise, let  $i_2$  be the individual for whom  $a(i_2) = b(i_1)$ . If  $b(i_2) = a(i_1)$ , add the trading cycle  $(i_1, i_2)$  to *T*. Otherwise let  $i_3$  be the individual for whom  $a(i_3) = b(i_2)$ , and continue in same way until an individual  $i_k$  is reached for whom  $b(i_k) = a(i_1)$ ; the number of individuals is finite, so such an individual exists. At this point, add the trading cycle  $(i_1, i_2, ..., i_k)$  to *T*.

If any individuals remain, select one of them arbitrarily and repeat the construction. Continue until every individual is a member of a trading cycle in *T*. By construction, *T* transforms *a* to *b* and is a trading partition because no individual appears in more than one of the trading cycles it contains. Given that for any individual *i*, the individual *j* for whom b(j) = a(i) is unique, *T* is the only trading partition that transforms *a* to *b*.

We now show that for any equilibrium allocation a the prices of all houses initially owned by the members of each trading cycle in the trading partition that transforms e to a are the same.

#### Proposition 9.1: Transforming initial allocation to equilibrium by trade

Let (p, a) be an equilibrium of the market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$ . The prices of all houses initially owned by the members of each trading cycle in the trading partition that transforms *e* to *a* are the same.

#### Proof

Let  $(i_1, ..., i_k)$  be a trading cycle in the trading partition that transforms e to a (described in the proof of Lemma 9.1). Then  $a(i_l) = e(i_{l+1})$  for

l = 1, ..., k - 1 and  $a(i_k) = e(i_1)$ . Thus for l = 1, ..., k - 1 we need  $e(i_{l+1})$  to be in  $i_l$ 's budget set, so that  $p(e(i_l)) \ge p(e(i_{l+1}))$ , and we need  $e(i_1)$  to be in  $i_k$ 's budget set, so that  $p(e(i_k)) \ge p(e(i_1))$ . Hence  $p(e(i_1)) \ge p(e(i_2)) \ge \cdots \ge p(e(i_k)) \ge p(e(i_1))$ , so that all these prices are equal.

# 9.2 Existence and construction of a market equilibrium

We now show that every market has an equilibrium. In fact, we show how to construct an equilibrium. The construction involves a sequence of trading cycles. We start by identifying a trading cycle that gives every individual in the cycle her favorite house. We call such a cycle a top trading cycle.

# **Definition 9.4: Top trading cycle**

The trading cycle  $(i_1, ..., i_k)$  in the market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$  is a *top trading cycle* if for l = 1, ..., k individual  $i_l$ 's favorite house is initially owned by individual  $i_{l+1}$ , where  $i_{k+1} = i_1$ . That is,  $e(i_{l+1}) \succeq^{i_l} h$  for l = 1, ..., k and all  $h \in H$ .

To find a top trading cycle, first choose an arbitrary individual, say  $i_1$ . If she initially owns her favorite house, then  $(i_1)$  is a (degenerate) top trading cycle. Otherwise, let  $i_2$  be the initial owner of  $i_1$ 's favorite house. If  $i_2$  initially owns her favorite house, then  $(i_2)$  is a top trading cycle; if  $i_1$  initially owns this house then  $(i_1, i_2)$  is a top trading cycle; otherwise let  $i_3$  be the owner. Continue in the same way, at each step k checking whether the owner of k's favorite house is a member of the sequence  $(i_1, \ldots, i_k)$ , say  $i_1$ , in which case  $(i_1, \ldots, i_k)$  is a top trading cycle, and otherwise adding the owner to the list as  $i_{k+1}$ . The number of individuals is finite, so eventually the procedure identifies a top trading cycle. The procedure is illustrated in the following diagram, in which an arrow from i to j means that j is the owner of i's favorite house, and then defined more formally.

# Procedure 9.1: Procedure for generating a top trading cycle

For a market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$ , the following inductive procedure generates a top trading cycle.

#### Initialization

Choose an arbitrary individual  $i_1 \in N$ , and define the sequence  $(i_1)$ .

#### **Inductive step**

Let  $(i_1, ..., i_k)$  be the sequence of individuals in N that is obtained in step k, so that  $e(i_{l+1})$  is  $i_l$ 's favorite house for l = 1, ..., k - 1.

- If the owner of *i<sub>k</sub>*'s favorite house is a member of the sequence, say *i<sub>l</sub>*, stop; the sequence (*i<sub>l</sub>*,...,*i<sub>k</sub>*) is a top trading cycle.
- Otherwise, add the owner of  $i_k$ 's favorite house to the sequence as  $i_{k+1}$ , to generate the sequence  $(i_1, \ldots, i_k, i_{k+1})$ , so that  $e(i_{l+1})$  is  $i_l$ 's favorite house for  $l = 1, \ldots, k$ , and continue.

Notice that the procedure is initialized with an arbitrary individual. The individual chosen may affect the top trading cycle that is generated. Consider, for example, a market in which individual *i* initially owns her favorite house, individual *j* owns the favorite house of individual *k*, and individual *k* owns the favorite house of individual *j*. Then the procedure generates the (degenerate) cycle (*i*) if we initially select individual *i* and the cycle (*j*, *k*) if we initially select individual *j*.

We now specify an iterative procedure that generates an equilibrium of a market. The procedure first finds a top trading cycle in the market, assigns the same arbitrary price, say  $p_1$ , to all the houses initially owned by individuals in the cycle, and assigns to each individual in the cycle her favorite house (that is, the house owned by the next individual in the cycle). It then removes all these individuals (and the houses they initially own) from the market, and finds a top trading cycle in the smaller market. It assigns an arbitrary price  $p_2$  with  $p_2 < p_1$  to the houses initially owned by the individuals in this cycle and assigns to each individual in the cycle her favorite house among those available in the smaller market. It then removes the individuals in the cycle from the smaller market, to produce an even smaller market. The procedure continues in the same way until no individuals remain.

# Procedure 9.2: Top trading procedure

For a market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$ , the *top trading procedure* is defined as follows. First, for any set  $N' \subseteq N$ , define M(N') be the market in which the set of individuals is N', the set H' consists of the houses owned initially by members of N', the preference relation of each member of N' is her original preference relation restricted to H', and the initial allocation assigns to each member of N' the house she owns in the original market.

#### Initialization

Start with the set of individuals  $N_1 = N$ , and any number  $p_0 > 0$ .

#### Inductive step

For a given set of individuals  $N_s \subseteq N$ , find a top trading cycle in the market  $M(N_s)$ ; denote by  $I_s$  the set of individuals in the cycle.

Assign a price  $p_s$  with  $0 < p_s < p_{s-1}$  to all the houses initially owned by the individuals in  $I_s$ , and assign to each member of  $I_s$  her favorite house in  $M(N_s)$  (the house initially owned by the individual who follows her in the top trading cycle).

Let  $N_{s+1} = N_s \setminus I_s$ . If  $N_{s+1} = \emptyset$ , stop; otherwise continue with  $N_{s+1}$ .

This procedure generates a price system and an assignment.

As we mentioned previously, a market may contain more than one top trading cycle, so different operations of the procedure may lead to different outcomes. We now show that every outcome of the procedure is an equilibrium of the market.

# Proposition 9.2: Existence of a market equilibrium

Every market has an equilibrium; any pair consisting of a price system and an assignment generated by the top trading procedure is an equilibrium.

#### Proof

The assignment and price system generated by the top trading procedure is an equilibrium because (1) every house is assigned only once, so that the assignment is an allocation, and (2) every individual is assigned her favorite house among all houses that are not more expensive than the house she owns initially.

# Example 9.6

For the market in Example 9.1, the top trading procedure operates as follows. The only top trading cycle in the entire market is (1,2). We assign a price  $p_1$  to the houses initially owned by individuals 1 and 2, *A* and *B*, allocate to each of these individuals her favorite house (a(1) = B and a(2) = A), and remove these individuals from the market.

The smaller market has two top trading cycles, (3) and (4). If we choose (3), then we assign a price  $p_2 < p_1$  to the house individual 3 initially owns, *C*, allocate this house to her, and remove her from the market.

Now only individual 4 remains, and in this market the only top trading cycle is (4). So we assign a price  $p_3 < p_2$  to the house individual 4 initially owns, *D*, and assign this house to her.

Thus the pair consisting of the allocation (B, A, C, D) and price system p with p(D) < p(C) < p(A) = p(B) is an equilibrium of the market.

If, at the second stage, we select individual 4 instead of individual 3, we generate the same allocation (B, A, C, D), but a price system p with p(C) < p(D) < p(A) = p(B), so such a pair is also an equilibrium of the market.

# Example 9.7

For the market in Example 9.2, the top trading procedure operates as follows. The only top trading cycle of the market is (1,4,2). We assign some price  $p_1$  to the houses initially owned by individuals 1, 2, and 4, allocate to each of these individuals her favorite house (a(1) = D, a(2) = A, and a(4) = B), and remove the individuals from the market.

The only individual remaining in the market is 3, and thus the only top trading cycle in the smaller market is (3). We assign a price  $p_2 < p_1$  to the house individual 3 initially owns, *C*, and assign this house to her.

Thus the pair consisting of the allocation (D, A, B, C) and price system p with p(C) < p(A) = p(B) = p(D) is an equilibrium of the market, as we saw in Example 9.2.

Proposition 9.2 does not assert that the equilibrium is unique, and indeed the ranking of prices may differ between equilibria, as Example 9.6 shows. However, in both examples the procedure finds only one equilibrium *allocation*, and this property is general: every market has a unique equilibrium allocation. We defer this result, Proposition 9.5, to a later section because its proof is somewhat more complex than the other proofs in this chapter.

Say that *i* is richer than *j* if p(e(i)) > p(e(j)), so that *i* can afford any house that *j* can afford and also at least one house that *j* cannot afford. What makes an individual in a market richer than other individuals? One factor that seems intuitively important in making *i* rich is the number of individuals whose favorite house is the house initially owned by *i*. The more individuals who like *i*'s house, the more likely *i* is to be a member of a top trading cycle that appears early in the procedure in the proof of Proposition 9.2, so that a high price is attached to her house. But the ranking of an individual's house by the other individuals is not the only factor in determining her wealth. The house owned initially by *i* may be the favorite house of many individuals, but those individuals may initially own houses that no one likes. In this case, although many individuals desire *i*'s house, *i* will not necessarily be relatively rich. Another contributor to a high market price for *i*'s house is the attractiveness of the houses owned by the individuals who like *i*'s house. Overall, it is the coordination of desires that makes a person rich in this model. For example, if for two individuals *i* and *j*, *i*'s favorite house is the one initially owned by *j*, and vice versa, and these houses are at the bottom of the rankings of all other individuals, the equilibrium price of the two houses could be higher than the equilibrium price of any other house.

# 9.3 Equilibrium and Pareto stability

Proposition 8.4, in the previous chapter, shows the Pareto stability of any equilibrium of a jungle without externalities (in which each individual cares only about the house she occupies and not about the house anyone else occupies). We now show an analogous result for a market. That is, if each individual cares only about the house she occupies, then for any equilibrium allocation in a market, no other allocation is at least as good for every individual and preferred by at least one individual.

#### Proposition 9.3: Pareto stability of equilibrium allocation

For any market, every equilibrium allocation is Pareto stable.

#### Proof

Let (p, a) be an equilibrium of the market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$ . If a is not Pareto stable then for some allocation b we have  $b(i) \succeq^i a(i)$  for every  $i \in N$  and  $b(i) \succ^i a(i)$  for some  $i \in N$ . For any i for which  $b(i) \succ^i a(i)$ we have p(b(i)) > p(a(i)), since otherwise  $p(b(i)) \le p(a(i)) = p(e(i))$  and thus a(i) is not optimal for i in the set  $\{h \in H : p(h) \le p(e(i))\}$ . For any other i, b(i) = a(i) (because each preference relation is strict) and thus p(b(i)) = p(a(i)). Hence  $\sum_{i \in N} p(b(i)) > \sum_{i \in N} p(a(i))$ . But a and b are both allocations, so each side of this inequality is equal to  $\sum_{i \in N} p(e(i))$ . This contradiction implies that no such allocation b exists, and hence a is Pareto stable.

The name conventionally given to this result and similar results for other models of economies is the "first fundamental theorem of welfare economics". However, the result establishes only that an equilibrium is Pareto stable, a concept unrelated to welfare. The concept of a market specifies only the individuals' ordinal preferences, not any measure of their welfare in the everyday sense of the word. The result says that for any equilibrium allocation, no other allocation exists for which some individual is better off and no individual is worse off. But allocations may exist in which the vast majority of individuals, and even all individuals but one, are better off. For these reasons, we refrain from using the conventional label for the result.

An implication of Proposition 9.3 is that if the initial allocation is not Pareto stable, every equilibrium involves trade: at least two individuals trade their houses. We now show conversely that if the initial allocation is Pareto stable then no trade occurs in equilibrium.

#### Proposition 9.4: No trade from a Pareto stable allocation

Let  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$  be a market. If *e* is Pareto stable then for every equilibrium (p, a) of the market we have a = e.

#### Proof

Let (p, a) be an equilibrium of the market. Since e(i) is in the budget set of individual *i* given the price system p,  $a(i) \succeq^i e(i)$  for all  $i \in N$ . If  $a \neq e$ then  $a(i) \succ^i e(i)$  for some individual *i*, which means that *e* is not Pareto stable.

A conclusion from Propositions 9.3 and 9.4 is that if a market starts operating at date 1 and results in an equilibrium allocation, and then is opened again at date 2 with initial endowments equal to the equilibrium allocation at the end of date 1, then no trade occurs in the equilibrium of the market at date 2.

Proposition 9.4 has an interpretation parallel to the one we give to Proposition 8.5 for a jungle. We interpret that result to mean that an authority in the society that controls the power relation and is aware of the individuals' preferences can obtain any Pareto stable allocation as an equilibrium by choosing the power relation appropriately. Proposition 9.4 may be given a similar interpretation. Suppose that an authority in the society can allocate initial property rights but cannot prevent individuals from trading. If trade is conducted according to the equilibrium concept we have defined, the authority can induce any Pareto stable allocation by assigning the initial rights appropriately. In this way, ownership in a market plays a role parallel to power in a jungle.

Proposition 9.4 is often called the "second fundamental theorem of welfare economics". We refrain from using this name because as for Proposition 9.3, we regard the word "welfare" as inappropriate in the context of the result.

#### 9.4 Uniqueness of market equilibrium

The uniqueness of equilibrium in an economic model is appealing because it means that the model narrows down the outcome as much as possible. It also

simplifies an analysis of the effect of a change in a parameter of the model.

The ranking of the prices in an equilibrium of a market is not necessarily unique, as Example 9.6 shows. But we now show that every market has a unique equilibrium *allocation*. This result depends on our assumption that no individual is indifferent between any two houses (see Problem 7).

### Proposition 9.5: Uniqueness of equilibrium

Every market has a unique equilibrium allocation.

#### Proof

Assume, contrary to the claim, that the market  $\langle N, H, (\succeq^i)_{i \in N}, e \rangle$  has equilibria (p, a) and (q, b) with  $a \neq b$ . Let  $i_1$  be an individual whose initial house  $e(i_1)$  has the highest price according to p. Let  $(i_1, i_2, ..., i_{k^*})$  be a trading cycle in the trading partition that transforms e into a. By Proposition 9.1, p(e(i)) is the same for all  $i \in I = \{i_1, i_2, ..., i_{k^*}\}$ , so that every house is in the budget set of every  $i \in I$ , and hence a(i) is i's favorite house in H for every  $i \in I$ .

Now consider the equilibrium (q, b). Without loss of generality,  $e(i_1)$  is the most expensive house according to q among the houses in  $\{e(i) : i \in I\}$ . That is,  $q(e(i_1)) \ge q(e(i))$  for all  $i \in I$ . Since  $e(i_2) = a(i_1)$  is  $i_1$ 's favorite house in H, in the equilibrium (q, b) individual  $i_1$  chooses  $e(i_2)$ . That is,  $b(i_1) = e(i_2)$ . Therefore, by Proposition 9.1,  $q(e(i_2)) = q(e(i_1))$ , so that  $i_2$ also owns a most expensive house according to q among the houses in  $\{e(i) : i \in I\}$ . Continue in this way to conclude that  $(i_1, i_2, ..., i_k)$  is also a trading cycle in the trading partition that transforms e to b.

Now delete from the set of individuals and the set of houses the members of this trading cycle and their initial houses. We are left with a smaller market and two equilibria of this market. The reason is that if, before the deletion, every individual chose the best house that she could afford given the equilibrium prices, the deletion of the houses that she did not choose does not affect the optimality of her choice. Therefore we can continue with the restricted market and choose again a trading cycle with the highest price according to p.

Continuing in this way, we conclude that the trading partition that transforms *e* into *a* is the same as the one that transforms *e* into *b*, so that a = b.

More formally, we can prove the result by induction on the number of individuals. A market with one individual of course has a unique equilibrium allocation. If every market with not more than n - 1 individuals has

a unique equilibrium allocation, then the argument we have made shows that a market with *n* individuals also has a unique equilibrium allocation.

# Problems

- 1. *Trade for an allocation that is not Pareto stable*. Show that if an allocation is not Pareto stable then some (nonempty) group of individuals can exchange the houses they own among themselves in such a way that all members of the group are better off.
- 2. Examples of markets.
  - *a.* Consider a market in which some individual initially owns her favorite house. Show that in any equilibrium this individual is allocated this house.
  - *b.* What can you say about the equilibrium allocation in a market equilibrium in which every house has a different price?
  - *c*. Show that in an equilibrium of any market consistent with the following table, individual 4 is allocated her favorite house.

Individual	1	2	3	4
Initial allocation	D	С	В	A
Favorite house	Α	Α	D	?

- 3. *Effect of removing an individual.* Give an example of a market for which removing one of the individuals, together with the house she initially owns, makes one of the remaining individuals better off and another of the remaining individuals worse off.
- 4. Effect of changes in one individual's preferences.
  - *a*. Let  $M_1$  be a market and let (p, a) be an equilibrium of  $M_1$ . Assume that  $M_2$  differs from  $M_1$  only in that a(1) moves up in individual 1's preferences. What can you say about the equilibrium allocations in the two markets?
  - *b.* (More difficult.) In the market  $M_1$ , individual 1 initially owns house *A*. The market  $M_2$  differs from  $M_1$  only in that in  $M_2$  the ranking of *A* in individual 2's preferences is higher than it is in  $M_1$ . Show that individual 1 is not worse off, and may be better off, in the equilibrium of  $M_2$  than in the equilibrium of  $M_1$ .

- 5. *Manipulation*. Explain why no individual in a market is better off behaving as if her preferences are different from her actual preferences. That is, if the markets *M* and *M'* differ only in the preferences of individual *i*, then the equilibrium allocation in *M'* is no better according to *i*'s preferences in *M* than the equilibrium allocation in *M*.
- 6. *The core*. Like Pareto stability, the core is a notion of stability. An allocation *a* is in the core of a market if no set of individuals can leave the market with their initial houses and reallocate them among themselves (in any way, not necessarily consistent with equilibrium) so that all of them are better off than in *a*. Show that the equilibrium of any market is in the core.
- 7. *Market with indifferences*. Some of the results in this chapter rely on the assumption that the individuals' preference relations do not have indifferences. Construct a market in which individuals have preferences with indifferences, some equilibrium is not Pareto stable, and there is more than one equilibrium allocation.

# Notes

The model presented in this chapter is due to Shapley and Scarf (1974), who attribute the proof for the existence of a market equilibrium to David Gale. The presentation here draws upon Rubinstein (2012, Chapter 3).