Models in Microeconomic Theory

Expanded Second Edition

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This book is available in two versions, one that uses feminine pronouns and one that uses masculine pronouns. This version uses feminine pronouns.

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10 An exchange economy

In this chapter we study a market in which the goods, unlike the houses in the previous two chapters, can be consumed in any quantity: they are divisible. As in the previous chapter, the ownership of goods is recognized and protected. Each individual initially owns a bundle of goods. We look for a distribution of the goods among the individuals and a price system with the property that for each good the total amount the individuals want to purchase is equal to the total amount other individuals want to sell: demand and supply are equal.

10.1 Model

In a market there are two goods, called 1 and 2. Each good can be consumed in any (nonnegative) quantity. As in Chapter 4, a bundle is a pair (x_1, x_2), where x_k, the quantity of good k for $k = 1, 2$, is a nonnegative number, so that the set of all possible bundles is $\mathbb{R}^2_+$. The set of individuals in the market is denoted N. Each individual $i \in N$ initially owns the bundle $e(i) = (e_1(i), e_2(i))$. We take these initial bundles as given; we do not ask where they come from. We assume that the total amount of each good initially owned by all individuals is positive (not zero).

Each individual cares about the bundle she owns after trading. Sometimes we say she "consumes" this bundle. As in the previous two chapters, we assume that each individual has no interest, selfish or altruistic, in the bundles chosen by other individuals. Thus the desires of each individual i are captured by a preference relation over the set $\mathbb{R}^2_+$ of possible bundles, which we assume is monotone and continuous.

Collecting these elements, we define an exchange economy as follows.

Definition 10.1: Exchange economy

An *exchange economy* $\langle N, (\succeq^i)_{i \in N}, e \rangle$ consists of

individuals
    a finite set N

preferences
    for each individual $i \in N$, a monotone and continuous preference relation $\succeq^i$ over $\mathbb{R}^2_+$

> **initial allocation**
> a function e that assigns to each individual $i \in N$ a bundle $e(i) \in \mathbb{R}^2_+$, the bundle that i initially owns, with $\sum_{i \in N} e_k(i) > 0$ for $k = 1, 2$.

Comments

1. The model of an exchange economy is closely related to that of a market discussed in the previous chapter. In the model of a market, each house is an indivisible good that can be consumed by only a single individual. In the model of an exchange economy, each good is divisible, and the total amount of it can be divided arbitrarily among the individuals. The analogue of the set H in the previous chapter is the set $\mathbb{R}^2_+$ here.

2. Many goods are in fact not divisible. For example, you can own four or six chairs, but not 5.3. We assume divisibility because it simplifies the analysis without, apparently, significantly affecting the conclusions.

3. Like the houses in the previous chapters, any given amount of each good in our model can be consumed by only one individual: the total amount of the good available has to be divided up among the individuals. This formulation excludes from consideration goods like information that can be simultaneously consumed by many individuals.

10.1.1   Prices and budget sets

A price system is a pair of nonnegative numbers $p = (p_1, p_2)$ different from $(0, 0)$. Given a price system p, the value of the bundle $x = (x_1, x_2)$ is $p_1 x_1 + p_2 x_2$, which we write also as px (the inner product of the vectors p and x). By exchanging some or all of her initial bundle $e(i)$ with other individuals, i can obtain any bundle x whose value px does not exceed $pe(i)$, the value of $e(i)$; that is, she can obtain any bundle x for which $px \leq pe(i)$. As before we refer to the set of such bundles as the budget set of individual i and denote it $B(p, e(i))$. Given our assumption that each individual's preference relation is monotone, a bundle is optimal in i's budget set if and only if it is optimal on the budget line $\{x \in \mathbb{R}^2_+ : px = pe(i)\}$.

> **Definition 10.2: Price system and budget set**
>
> A *price system* is a pair of nonnegative numbers different from $(0, 0)$. The *value of the bundle $x = (x_1, x_2)$ according to the price system $p = (p_1, p_2)$ is*

$px = p_1 x_1 + p_2 x_2$, and for the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$, the *budget set of individual* $i \in N$ given the price system p is

$$B(p, e(i)) = \{x \in \mathbb{R}^2_+ : px \leq pe(i)\}.$$

We have in mind two interpretations of a price system. The first and most literal is that the prices are quoted in a monetary unit. Each individual can sell any amounts of the goods in her initial bundle and use the monetary proceeds to buy amounts of other goods. If, for example, she sells y_1 units of good 1 then she obtains the amount of money $p_1 y_1$, which she can use to buy the amount z_2 of good 2 for which $p_1 y_1 = p_2 z_2$. The second interpretation is that the prices represent the ratio at which the goods may be exchanged. Specifically, the price system (p_1, p_2) means that one unit of good 1 may be exchanged for p_1/p_2 units of good 2.

Note that in both interpretations a price system p is equivalent to any price system of the form $\lambda p = (\lambda p_1, \lambda p_2)$ for $\lambda > 0$ (that is, a price system in which all prices are multiplied by a positive number), because $B(\lambda p, e(i)) = B(p, e(i))$ for all values of p and $e(i)$.

10.1.2 Allocations

The total amount of each good k available in the economy is $\sum_{i \in N} e_k(i)$. An *allocation* is a distribution of these total amounts among the individuals.

Definition 10.3: Assignment and allocation

An *assignment* in an exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ is a function from the set N of individuals to the set $\mathbb{R}^2_+$ of possible bundles of goods. An *allocation* is an assignment a for which the sum of the assigned bundles is the sum of the initial bundles:

$$\sum_{i \in N} a(i) = \sum_{i \in N} e(i).$$

10.2 Competitive equilibrium

The central concept in this chapter is competitive equilibrium. A competitive equilibrium consists of a price system and an assignment such that each individual's bundle in the assignment is optimal for her given the price system and her initial endowment, and the assignment is an allocation. If this condition is

not satisfied then either at least one individual does not choose her optimal bundle or, for at least one of the goods, the total amount that the individuals want to purchase is different from the total amount available.

Definition 10.4: Competitive equilibrium of exchange economy

A *competitive equilibrium* of the exchange economy $\langle N,(\succeq^i)_{i\in N}, e\rangle$ is a pair (p, a) in which

- $p = (p_1, p_2)$ is a price system

- a is an assignment

such that

optimality of choices
  for every individual $i \in N$ the bundle $a(i)$ is optimal according to $\succeq^i$ in the budget set $B(p, e(i))$ (that is, $B(p, e(i))$ contains no bundle b for which $b \succ^i a(i)$)

feasibility
  a is an allocation.

  An allocation a is a *competitive equilibrium allocation* of the exchange economy $\langle N,(\succeq^i)_{i\in N}, e\rangle$ if for some price system p, (p, a) is a competitive equilibrium of $\langle N,(\succeq^i)_{i\in N}, e\rangle$.

Note that if (p, a) is a competitive equilibrium of $\langle N,(\succeq^i)_{i\in N}, e\rangle$ then so is $(\lambda p, a)$ for any $\lambda > 0$, because $B(\lambda p, e(i)) = B(p, e(i))$ for all $i \in N$.

In a competitive equilibrium all individuals face the same price system and each individual chooses an optimal bundle from a budget set defined by this price system, which is not affected by the individual's choice. This assumption seems reasonable when the market contains a large number of individuals, none of whom initially owns a large fraction of the total amount of any good. It is less reasonable when the number of individuals is small, in which case some individuals' actions may significantly affect the prices. Note, however, that the concept of competitive equilibrium is well-defined regardless of the number of individuals and the distribution of their initial bundles. In particular, it is well-defined even for an economy with only one individual; only the reasonableness of the concept is questionable in this case.

Figure 10.1 illustrates a competitive equilibrium for an exchange economy with two individuals. In this figure, the orange vectors are equal in length and opposite in direction, so that the sum of the individuals' optimal bundles is equal to the sum of their initial bundles.
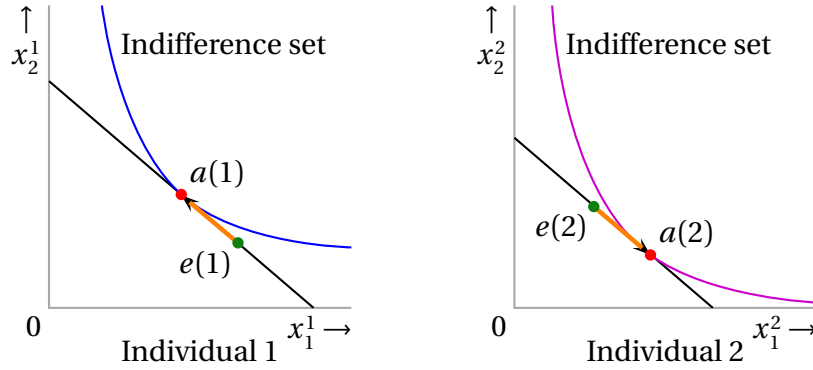
Figure 10.1 A competitive equilibrium in an exchange economy with two individuals. The ratio p_1/p_2 of the prices is the (common) slope of the (black) budget frontiers, and $a(1)$ and $a(2)$ are the bundles the individuals optimally choose. The prices are consistent with a competitive equilibrium because the orange vectors exactly cancel each other out: $e(1) - a(1) = -(e(2) - a(2))$, so that $a(1) + a(2) = e(1) + e(2)$.

Example 10.1: Competitive equilibrium with substitutable goods

Consider an exchange economy $\langle N, (\succcurlyeq^i)_{i \in N}, e \rangle$ for which $N = \{1, 2\}$, $e(1) = (\alpha, 0)$, and $e(2) = (0, \beta)$ (with $\alpha > 0$ and $\beta > 0$), so that each good is initially owned exclusively by one individual, and each individual's preference relation $\succcurlyeq^i$ is represented by the utility function $x_1 + x_2$ (Example 4.1 with $v_1/v_2 = 1$).

In this economy, (p, a) with $p = (1, 1)$ and $a = e$ is a competitive equilibrium. In this equilibrium, the budget lines of individuals 1 and 2 are $\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = \alpha\}$ and $\{(x_1, x_2) \in \mathbb{R}_+^2 : x_1 + x_2 = \beta\}$, so that each individual i is indifferent between all bundles on her budget line and the bundle $e(i)$, in particular, is optimal for her (see Example 5.6). Note that the concept of competitive equilibrium requires only that the bundle assigned to each individual is optimal for the individual, not that it is the *only* bundle optimal for her.

More generally, every pair (p, a) where $p = (1, 1)$, $a(1) = (\alpha - \varepsilon, \varepsilon)$, and $a(2) = (\varepsilon, \beta - \varepsilon)$, with $0 \le \varepsilon \le \min\{\alpha, \beta\}$, is a competitive equilibrium. In such an equilibrium each individual exchanges ε units of the good she initially owns for ε units of the other good. Neither individual can do better because given the price system, for every bundle an individual can achieve by exchange, the sum of the amounts of the two goods is the same.

This economy has no equilibrium in which the prices of the goods are not equal. For any such price system, each individual's unique optimal bundle contains none of the more expensive good and thus is not consistent with a competitive equilibrium.

Example 10.2: Competitive equilibrium with complementary goods

Consider an exchange economy $\langle N,(\succeq^i)_{i \in N}, e \rangle$ for which $N = \{1, 2\}$, $e(1) = (\alpha, 0)$, and $e(2) = (0, \beta)$, as in the previous example, but each individual's preference relation is represented by the utility function $\min\{x_1, x_2\}$ (Example 4.4).

For any price system p in which $p_1 > 0$ and $p_2 > 0$, the bundles optimally chosen by the individuals are

$$x^1(p) = \left(\frac{\alpha p_1}{p_1 + p_2}, \frac{\alpha p_1}{p_1 + p_2} \right) \quad \text{and} \quad x^2(p) = \left(\frac{\beta p_2}{p_1 + p_2}, \frac{\beta p_2}{p_1 + p_2} \right)$$

(see Example 5.5). For this pair of bundles to be an allocation we need

$$\frac{\alpha p_1}{p_1 + p_2} + \frac{\beta p_2}{p_1 + p_2} = \alpha \quad \text{and} \quad \frac{\alpha p_1}{p_1 + p_2} + \frac{\beta p_2}{p_1 + p_2} = \beta.$$

The left-hand sides of these equations are the same, so that if $\alpha = \beta$ then for every price system p with $p_1 > 0$ and $p_2 > 0$ the economy has a competitive equilibrium (p, a) with $a(1) = x^1(p)$ and $a(2) = x^2(p)$.

If $\alpha \neq \beta$ then no equilibrium in which both prices are positive exists, but the economy has an equilibrium in which one of the prices is zero. Suppose that $\alpha > \beta$. Then the economy has an equilibrium in which the price system is $(0, 1)$ (good 1, of which there is a surplus, has no value). Given that the price of good 1 is zero, individual 2 can consume any quantity of good 1, so that any bundle (x_1^2, β) with $x_1^2 \geq \beta$ is optimal for her. Individual 1, who has only good 1, is indifferent between all bundles $(x_1^1, 0)$ on her budget line. For a competitive equilibrium we need $x_1^1 + x_1^2 = \alpha$, so $((0, 1), a)$ is a competitive equilibrium if and only if $a(1) = (x_1^1, 0)$ and $a(2) = (\alpha - x_1^1, \beta)$, with $x_1^1 \leq \alpha - \beta$. In particular, $((0, 1), a)$ with $a(1) = (\alpha - \beta, 0)$ and $a(2) = (\beta, \beta)$ is a competitive equilibrium.

This example shows, incidentally, that an individual who destroys some of her initial bundle may *improve* the bundle she consumes in a competitive equilibrium. If $\alpha > \beta$, then the bundle individual 1 consumes in a competitive equilibrium is $(x_1^1, 0)$ where $x_1^1 \leq \alpha - \beta$. If she destroys some of her initial holding of good 1, reducing the amount to γ with $0 < \gamma < \beta$, then the competitive equilibrium price system changes from $(0, 1)$ to $(1, 0)$ and the equilibrium allocations a have $a(1) = (\gamma, \beta - x_2^2)$ and $a(2) = (0, x_2^2)$ for $0 \leq x_2^2 \leq \beta - \gamma$. Individual 1 prefers all of these allocations to the ones in the original equilibrium. The fact that an

individual may benefit from destroying some of the goods she initially owns does not depend on an equilibrium price being zero; other examples show that the phenomenon may occur when both equilibrium prices are positive.

   The definition of a competitive equilibrium requires that the sum of the bundles optimal for the individuals, given the prices, is equal to the sum of the initial bundles (demand is equal to supply for all goods). The next result says that given any price system and any assignment that consists of optimal bundles (given the price system), if for one good the sum of the assigned quantities is equal to the sum of the individuals' initial holdings, then the same is true also for the other good. This result is useful when calculating competitive equilibria, because it means that if we find a price system for which demand and supply are equal for one good then we know that they are equal for the other good, so that the price system is consistent with competitive equilibrium.

Proposition 10.1: Property of assignment of bundles on budget lines

Consider an exchange economy $\langle N, (\succsim^i)_{i \in N}, e \rangle$. Let p be a price system with $p_1 > 0$ and $p_2 > 0$. Consider an assignment a with $pa(i) = pe(i)$ for all $i \in N$. (That is, $a(i)$ is on i's budget line for each i.) If the sum of the quantities of one good in the bundles in the assignment a is equal to the sum of the quantities of the good in the initial bundles then this equality holds also for the other good. That is,

$$\sum_{i \in N} a_1(i) = \sum_{i \in N} e_1(i) \quad \Leftrightarrow \quad \sum_{i \in N} a_2(i) = \sum_{i \in N} e_2(i).$$

Proof

The fact that $pa(i) = pe(i)$ for each $i \in N$ means that $p_1 a_1(i) + p_2 a_2(i) = p_1 e_1(i) + p_2 e_2(i)$ for each $i \in N$, and hence

$$\sum_{i \in N} \left[p_1 a_1(i) + p_2 a_2(i) \right] = \sum_{i \in N} \left[p_1 e_1(i) + p_2 e_2(i) \right].$$

Thus

$$p_1 \left(\sum_{i \in N} a_1(i) - \sum_{i \in N} e_1(i) \right) = p_2 \left(\sum_{i \in N} e_2(i) - \sum_{i \in N} a_2(i) \right),$$

so that given $p_1 > 0$ and $p_2 > 0$,

$$\sum_{i \in N} a_2(i) - \sum_{i \in N} e_2(i) = 0 \quad \Leftrightarrow \quad \sum_{i \in N} e_1(i) - \sum_{i \in N} a_1(i) = 0.$$

> ### Example 10.3
>
> Consider an exchange economy $\langle N,(\succsim^i)_{i\in N}, e\rangle$ in which, as in the previous
> two examples, $N = \{1,2\}$, $e(1) = (\alpha, 0)$, and $e(2) = (0, \beta)$. Assume now that
> each individual's preference relation is represented by the utility function
> $x_1 x_2$. Recall that the optimal bundle for such an individual has the prop-
> erty that the amount the individual spends on each good is the same (see
> Example 5.7). Now, if $((p_1, 1), a)$ is a competitive equilibrium with $p_1 > 0$,
> then individual 1 spends $p_1\alpha/2$ on each good and thus $a(1) = (\alpha/2, p_1\alpha/2)$.
> Similarly $a(2) = (\beta/(2p_1), \beta/2)$. By Proposition 10.1, $((p_1, 1), a)$ is a com-
> petitive equilibrium if and only if $\alpha/2 + \beta/(2p_1) = \alpha$. The economy has
> no equilibrium in which a price is zero, so (p, a) with $p = (\beta/\alpha, 1)$ and
> $a(1) = a(2) = (\alpha/2, \beta/2)$ is the only equilibrium.

10.3   Existence of a competitive equilibrium

A result that gives precise conditions under which an exchange economy has
a competitive equilibrium requires mathematical tools beyond the level of this
book. However, we can establish the following result, for the case in which each
individual has a continuous demand function. When each individual is ratio-
nal and has strictly convex and continuous preferences, her optimal choice as a
function of the price system results in such a demand function. The result states
that if every individual wants to obtain more of a good than she initially owns
when its price is low enough and sell some of her initial holding when its price is
high enough, then a competitive equilibrium exists.

> ### Proposition 10.2: Existence of competitive equilibrium
>
> Let $\langle N,(\succsim^i)_{i\in N}, e\rangle$ be an exchange economy. For each $i \in N$ and every price
> system p with $p_1 > 0$ and $p_2 > 0$, let $x^i(p)$ be a bundle that maximizes $\succsim^i$
> in the budget set $B(p, e(i))$. Let $d^i(p_1) = x_1^i(B((p_1, 1), e(i)))$, individual i's
> demand for good 1 given the price system $(p_1, 1)$. Assume that
>
> - each function d^i is continuous
>
> - for some price p_1 low enough we have $d^i(p_1) > e_1(i)$ for all $i \in N$ (every
>   individual wants to consume more of good 1 than she initially owns)
>
> - for some price p_1 high enough we have $d^i(p_1) < e_1(i)$ for all $i \in N$
>   (every individual wants to sell some of the amount of good 1 that she
>   initially owns).
>
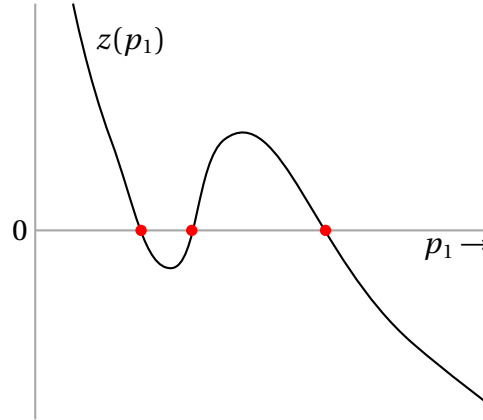> Then the exchange economy has a competitive equilibrium.

Figure 10.2 An example of an excess demand function z for good 1 satisfying the conditions of Proposition 10.2. The three red disks indicate competitive equilibrium prices for good 1.

Proof

For any price p_1, let $z(p_1) = \sum_{i \in N}[d^i(p_1) - e_1(i)]$, the difference between the total demand for good 1 when the price system is $(p_1, 1)$ and the total amount of good 1 available. By the assumption that each function d^i is continuous, the function z is continuous. By the assumptions about the values of the demand functions for low and high values of p_1, $z(p_1)$ is positive for p_1 small enough and negative for p_1 high enough. (A function z satisfying these conditions is shown in Figure 10.2.) By the Intermediate Value Theorem the value of this function z is thus zero for at least one price p_1^*. (It may be zero for more than one price.)

We claim that $((p_1^*, 1), (x^i(p_1^*, 1))_{i \in N})$ is a competitive equilibrium. The optimality condition is satisfied by the definition of the demand functions. The feasibility condition follows from Proposition 10.1, which states that if the excess demand for one good is zero, so is the excess demand for the second good.

An exchange economy in which some individuals' preferences are not convex may not have a competitive equilibrium; Problem 5 asks you to study an example. Example 10.2 shows that an exchange economy may have multiple equilibria, differing both in the price system and the equilibrium allocation.

10.4 Reopening trade

Consider an exchange economy $\langle N, (\succsim^i)_{i \in N}, e \rangle$ in which each individual i initially holds the bundle $e(i)$. Suppose that the individuals trade according to a

competitive equilibrium (p, a). After trade, each individual i holds the bundle $a(i)$. Now suppose that the possibility of trade reopens; the exchange economy $\langle N, (\succeq^i)_{i \in N}, a \rangle$, in which the initial bundle of each individual i is $a(i)$, models the situation. Does (p, a) remain a competitive equilibrium in this economy? The next result states that it does.

Proposition 10.3: No trade from competitive equilibrium

A competitive equilibrium (p, a) of the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ is a competitive equilibrium of the exchange economy $\langle N, (\succeq^i)_{i \in N}, a \rangle$.

Proof

The feasibility condition for competitive equilibrium is satisfied because a by definition is an allocation. The optimality condition is satisfied also: since $pa(i) = pe(i)$, the budget sets $B(p, e(i))$ and $B(p, a(i))$ are the same, so that the bundle $a(i)$, which is optimal in the budget set $B(p, e(i))$, is optimal for individual i also in $B(p, a(i))$.

10.5   Equilibrium and Pareto stability

We now show that a competitive equilibrium allocation is Pareto stable.

Proposition 10.4: Pareto stability of competitive equilibrium allocation

Every competitive equilibrium allocation of an exchange economy is Pareto stable.

Proof

Let (p, a) be a competitive equilibrium of the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$. Assume that a is not Pareto stable. That is, assume that there is an allocation y such that $y(i) \succeq^i a(i)$ for every individual i and $y(j) \succ^j a(j)$ for some individual j.

The optimality of $a(i)$ according to $\succeq^i$ in i's budget set implies that $py(i) \geq pe(i)$: if $py(i) < pe(i)$ then the budget set contains a bundle that i prefers to $y(i)$ and hence to $a(i)$. Furthermore, the optimality of $a(j)$ in j's budget set implies that $py(j) > pe(j)$. Thus $p \sum_{i \in N} y(i) > p \sum_{i \in N} e(i)$, contradicting the feasibility of y, which requires $\sum_{i \in N} y(i) = \sum_{i \in N} e(i)$. Hence a is Pareto stable.

Comments

1. The conclusion of this result depends critically on the assumption that each individual cares only about the bundle she consumes. Consider a variant of an exchange economy in which individuals care also about the bundles consumed by other individuals. Suppose specifically that the economy contains two individuals, with initial bundles $(1,0)$ and $(0,1)$. Individual 1 is negatively affected by individual 2's consumption of good 2; her utility from any allocation a is $a_1(1) + a_2(1) - 2a_2(2)$. Individual 2 cares only about her own consumption; her utility from a is $2a_1(2) + 3a_2(2)$. Assuming that each individual takes the consumption of the other individual as given, the only price systems $(p_1, 1)$ for which the demands for each good is equal to 1 satisfy $\frac{2}{3} \leq p_1 \leq 1$ and no trade occurs (the induced allocation is the initial allocation). This outcome is not Pareto stable: the allocation b for which $b(1) = (0, 0.5)$ and $b(2) = (1, 0.5)$ is preferred by both individuals.

2. An allocation is Pareto stable if no other allocation is at least as good for all individuals and better for at least one. We suggest that you verify that under either of the following conditions, an allocation is Pareto stable if and only if no other allocation is better for every individual.

    a. The individuals' preference relations are convex and every bundle in the allocation contains a positive quantity of each good.

    b. The individuals' preference relations are strongly monotone.

3. An implication of Proposition 10.4 is an analogue of Proposition 9.4. If the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ has a competitive equilibrium (p, a), then $a(i) \succeq^i e(i)$ for every individual $i \in N$. Thus if the initial allocation e is Pareto stable, then every individual i is indifferent between $a(i)$ and $e(i)$, so that (p, e) is also a competitive equilibrium of the economy. Suppose an authority that is able to redistribute goods between individuals wants the allocation in the economy to be some Pareto stable allocation a. The result says that if the authority redistributes goods to generate a, subsequently opening up trade will not undo the redistribution, in the sense that the outcome will be an allocation b for which $b(i) \sim^i a(i)$ for every individual i.

4. Proposition 10.4 is an analogue of Proposition 9.3. Like that result, it is conventionally referred to as the "first fundamental theorem of welfare economics". The result in the previous comment is an analogue of Proposition 9.4 and is conventionally referred to as the "second fundamental theorem of welfare economics". For the reasons we give in the discussion following Proposition 9.3, we regard these names as inappropriate.

10.6   The core

Proposition 10.4 says that no allocation is unanimously preferred to a competitive equilibrium allocation. In fact, a stronger result holds: for any competitive equilibrium allocation a, no group of individuals can benefit from seceding from the economy and reallocating their initial bundles among themselves (without exchanging goods with any individuals outside the group), in such a way that they are all better off than they are in a. To state the result, we use a stability concept called the core.

Definition 10.5: Core

Consider the exchange economy $\langle N, (\succsim^i)_{i \in N}, e \rangle$. A nonempty set $S \subseteq N$ of individuals can *improve upon* an allocation a if for some collection $(b(i))_{i \in S}$ of bundles with $\sum_{i \in S} b(i) = \sum_{i \in S} e(i)$ we have $b(i) \succ^i a(i)$ for all $i \in S$. An allocation a is in the *core* if no set of individuals can improve upon it.

Note that whether an allocation is in the core, unlike its Pareto stability, depends on the allocation of the initial bundles.

The following example shows that an allocation can be Pareto stable and preferred by every individual to his initial bundle and yet not be in the core.

Example 10.4: Pareto stable allocation not in core

Consider an exchange economy with two individuals of type 1 and two of type 2. Each individual of type 1 has the initial bundle $(1,0)$ and a preference relation represented by the utility function $\min\{x_1, x_2\}$. Each individual of type 2 has the initial bundle $(0,1)$ and a preference relation represented by the utility function $x_1 + x_2$.

Consider the allocation a in which each individual of type 1 is assigned the bundle $(0.1, 0.1)$ and each individual of type 2 is assigned the bundle $(0.9, 0.9)$. Each individual prefers her assigned bundle in this allocation to her initial bundle. The allocation is Pareto stable (for every bundle preferred by any individual, the sum of the amounts of the goods exceeds the sum of the amounts she is allocated). However, all members of a set S consisting of two individuals of type 1 and one of type 2, can improve upon the allocation: if each individual of type 1 is assigned the bundle $(0.2, 0.2)$ and the individual of type 2 is assigned the bundle $(1.6, 0.6)$, then all three individuals are better off than they are in the original allocation a.

> ### Proposition 10.5: Competitive equilibrium is in core
>
> Every competitive equilibrium allocation of an exchange economy is in the core of the economy.

> ### Proof
>
> Let (p, a) be a competitive equilibrium of the exchange economy $\langle N, (\succsim^i)_{i \in N}, e \rangle$. Suppose that a is not in the core of the economy. Then there is a nonempty set $S \subseteq N$ and a collection $(b(i))_{i \in S}$ of bundles with $\sum_{i \in S} b(i) = \sum_{i \in S} e(i)$ such that $b(i) \succ^i a(i)$ for all $i \in S$. Now, the fact that (p, a) is a competitive equilibrium means that for each individual i, the bundle $a(i)$ is optimal according to $\succsim^i$ in i's budget set. Thus $b(i) \succ^i a(i)$ implies that $pb(i) > pe(i)$, so that $p \sum_{i \in S} b(i) > p \sum_{i \in S} e(i)$. This inequality contradicts the condition $\sum_{i \in S} b(i) = \sum_{i \in S} e(i)$. Thus a is in fact in the core.

10.7 Competitive equilibrium based on demand functions

The individuals in an exchange economy are characterized by their preference relations and initial bundles. In a competitive equilibrium, each individual chooses her favorite bundle, according to her preferences, from her budget set. In a variant of the model, individuals are characterized instead by their demand functions and initial bundles, with the demand function of each individual specifying the bundle she consumes for each price system, given her initial bundle. These demand functions may not be rationalized by preference relations. (See Section 5.5 for examples of such demand functions.)

A competitive equilibrium of this variant of an exchange economy is a pair (p, a) consisting of a price system p and an assignment a such that (1) for each individual i the bundle $a(i)$ is the one specified by her demand function given the price system p and her initial bundle $e(i)$ and (2) a is an allocation.

> ### Example 10.5: Competitive equilibrium based on demand functions
>
> Consider an economy with two individuals in which individual 1 consumes only the good with the higher price; if the prices of the goods are the same, she consumes only good 1. No monotone preference relation rationalizes this demand function (see Example 5.11). Assume that individual 2 demands a bundle that maximizes the function $x_1 + x_2$ over her

budget set, so that if the prices of the goods differ, she demands only the good with the lower price.

Suppose that $e(1) = (\alpha, 0)$ and $e(2) = (0, \beta)$. Then every pair (p, a) in which p is a price system with $p_1 \geq p_2 = 1$ and a is the allocation with $a(1) = (\alpha, 0)$ and $a(2) = (0, \beta)$ is a competitive equilibrium since individual 1 demands only the first (and more expensive) good and individual 2 demands only the second (and cheaper) good.

For (p, a) with $p_1 < p_2 = 1$ to be an equilibrium we need $a(1) = (0, \alpha p_1)$ and $a(2) = (\beta/p_1, 0)$. These two bundles sum to the total bundle (α, β) if and only if $p_1 = \beta/\alpha$. Thus for such an equilibrium to exist we need $\beta < \alpha$. Indeed, if $\beta < \alpha$ then in addition to the equilibria in the previous paragraph, the pair (p, a) in which $p = (\beta/\alpha, 1)$, $a(1) = (0, \beta)$, and $a(2) = (\alpha, 0)$ is a competitive equilibrium.

10.8   Manipulability

Can an individual bias a competitive equilibrium in her favor by acting as if her preferences differ from her true preferences? The next example shows that the answer to this question is affirmative. (By contrast, Problem 5 in Chapter 9 shows that such manipulation is not possible in the markets studied in that chapter.)

Example 10.6: Manipulability of competitive equilibrium

Consider an exchange economy with two individuals in which the preference relation of individual 1 is represented by the function $x_1 + x_2$, the preference relation of individual 2 is represented by the function $\min\{x_1, x_2\}$, and the initial bundles are $e(1) = (1, 0)$ and $e(2) = (0, 1)$. This economy has a unique competitive equilibrium (p, a), with $p = (1, 1)$ and $a(1) = a(2) = (\frac{1}{2}, \frac{1}{2})$.

If individual 1 acts as if her preferences are represented by the function $3x_1 + x_2$ (which means that she acts as if good 1 is more desirable to her than it really is) then the competitive equilibrium prices are $(3, 1)$ and the equilibrium allocation gives her the bundle $(\frac{3}{4}, \frac{3}{4})$, which is better for her than the bundle $(\frac{1}{2}, \frac{1}{2})$ she receives in the equilibrium when she acts according to her true preferences.

10.9   Edgeworth box

The Edgeworth box is a graphical tool for analyzing an exchange economy with two individuals and two goods. We take a diagram like the one in Figure 10.1
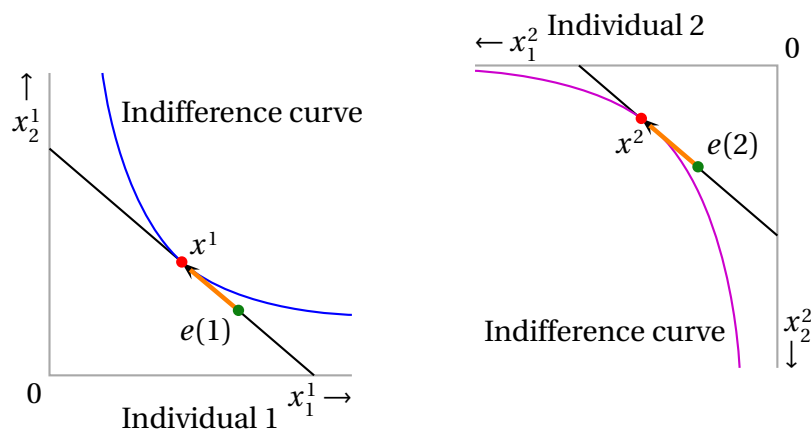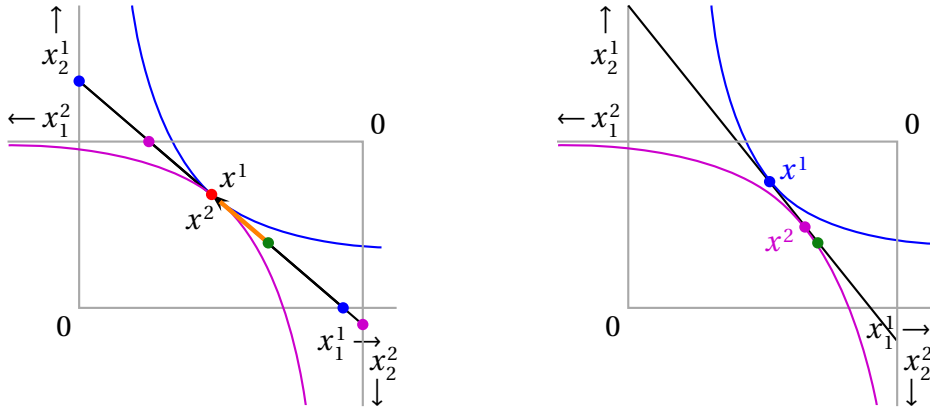
Figure 10.3 The competitive equilibrium in Figure 10.1, with the diagram for individual 2 rotated 180 degrees.

and rotate the right-hand panel, which represents individual 2's optimization problem, 180 degrees, to get Figure 10.3. Then we move the panels together, so that the initial bundles (represented by small green disks) coincide, to get Figure 10.4a. In this diagram, a point in the rectangle bounded by the two sets of axes represents an allocation, with the bundle assigned to individual 1 plotted relative to the axes with origin at the bottom left, and the bundle assigned to individual 2 plotted relative to the axes with origin at the top right.

The green disk in the figure represents the individuals' initial bundles; the segment of the black line between the blue disks (viewed relative to individual 1's axes) represents individual 1's budget set, and the segment between the violet disks (viewed relative to individual 2's axes) represents individual 2's budget set. The line corresponds to a competitive equilibrium if the disk representing the optimal bundle x^1 of individual 1 coincides with the disk representing the optimal bundle x^2 for individual 2, as it does in Figure 10.4a, because in this case the assignment in which each individual gets her optimal bundle given the price system is an allocation. Figure 10.4b shows a price system that does not correspond to a competitive equilibrium: the total amount of good 1 demanded is less than the total amount available and the total amount of good 2 demanded exceeds the total amount available.

The set of Pareto stable allocations and the core are shown in Figure 10.5. Every allocation on the line colored black and red connecting individual 1's origin to individual 2's origin is Pareto stable. The reason is that every allocation on or above the indifference curve of one individual through the allocation is below (relative to other individual's origin) the indifference curve of the other individual through the allocation. Similarly, any allocation not on the black and red line is not Pareto stable, because there is another allocation in which both individuals are better off.

(a) The competitive equilibrium in Figure 10.1, with the diagram for individual 2 rotated 180 degrees and moved so that the points representing the individuals' initial bundles coincide.

(b) A price system that is not consistent with a competitive equilibrium.

Figure 10.4 Edgeworth boxes.

An allocation is in the core of this two-individual economy if and only if it is Pareto stable and each individual likes the allocation at least as much as her initial bundle. Thus the core is the set of allocations on the red line in Figure 10.5.

Problems

1. *Examples.* Consider the following exchange economies, in which n^A individuals have preferences represented by the utility function u^A and initial bundle e^A and n^B individuals have preferences represented by the utility function u^B and initial bundle e^B.

economy	n^A	n^B	u^A	u^B	e^A	e^B
E_1	1	1	$x_1 + x_2$	$\min\{x_1, x_2\}$	$(\alpha, 0)$	$(0, \beta)$
E_2	1	2	$x_1 + x_2$	$\min\{x_1, x_2\}$	$(1, 0)$	$(0, 1)$
E_3	n^A	n^B	x_2	x_1	$(\alpha, 0)$	$(0, \beta)$
E_4	1	1	$x_1 x_2$	$x_1^2 x_2$	$(2, 3)$	$(5, 4)$

   a. Characterize the competitive equilibria of E_1 for $\alpha \leq \beta \leq 2\alpha$. Draw the Edgeworth box of E_1 and indicate the set of Pareto stable allocations and the core.

   b. Characterize the competitive equilibria of E_2.

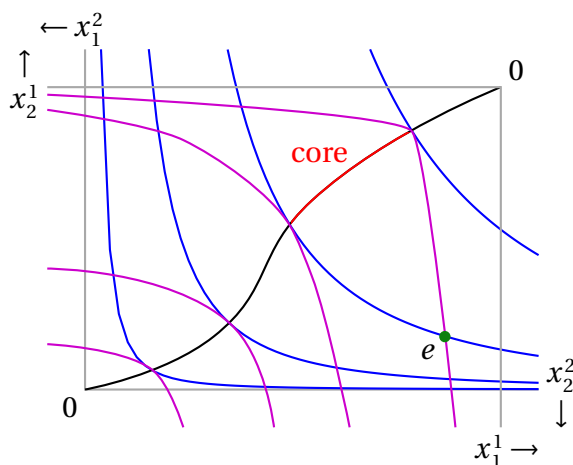   *c.* Characterize the competitive equilibria of E_3.

Figure 10.5 The core (red line) and set of Pareto stable allocations (core plus black lines).

    d. Find the competitive equilibria of E_4. (Ariel does not like problems like this one, but suggests you do the problem, so that you appreciate what you would be missing.) You should conclude that the competitive price system is $(\frac{25}{16}, 1)$.

2. *A market with perfectly complementary goods.* All individuals in an exchange economy have preferences represented by the function $\min\{x_1, x_2\}$; n_1 of them have the initial bundle $(1,0)$ and the remaining n_2 have the initial bundle $(0,1)$, where $n_1 \geq n_2$.

    a. Show that the allocation in which each individual holds the bundle $(n_1/(n_1 + n_2), n_2/(n_1 + n_2))$ is Pareto stable.

    b. Show that this allocation is in the core if and only if $n_1 = n_2$.

    c. What are the competitive equilibria of the economy when $n_1 > n_2$?

3. *Replicating a market.* Let $M_1 = \langle N, (\succcurlyeq^i)_{i \in N}, e \rangle$ be an exchange economy in which $N = \{A, B\}$. Let M_n be the exchange economy containing n individuals identical to A (type A) and n individuals identical to B (type B).

    a. Suppose that (p, a) is a competitive equilibrium of M_1 and that the assignment b in M_n gives each of the n individuals of type A the bundle $a(A)$ and each of the n individuals of type B the bundle $a(B)$. Show that (p, b) is a competitive equilibrium of M_n.

    b. Show that if all individuals have strictly convex preferences and (p, b) is a competitive equilibrium of M_n, then all individuals of type A consume the same bundle, say x_A, and all individuals of type B consume the same

bundle, say x_B, and (p, a) with $a(A) = x_A$ and $a(B) = x_B$ is a competitive equilibrium of M_1.

4. *Robinson Crusoe economy.* Consider an exchange economy with a single individual, R, who has the initial bundle e. A competitive equilibrium of this economy is a pair (p, x^*) where p is a price system and x^* is a bundle, with x^* optimal for R in $\{x \in \mathbb{R}^2_+ : px = pe\}$ and $x^* = e$. Assume that R's preference relation is monotone, continuous, and convex. Explain graphically why this economy has a competitive equilibrium.

5. *Economy with nonconvex preferences.* We remark before Proposition 10.2 that one of the sufficient conditions for the existence of a competitive equilibrium in an exchange economy is that the individuals' preferences are convex. In this question you will see that competitive equilibrium may exist if the individuals' preferences are not convex. Consider an exchange economy with two individuals whose preferences are represented by the utility function $(x_1)^2 + (x_2)^2$, and thus are not convex (see Problem 1a in Chapter 4). Assume that $e_k(1) + e_k(2) > 0$ for $k = 1, 2$.

   a. Show that the economy with $e(1) = (\alpha, 0)$ and $e(2) = (0, \beta)$ has a competitive equilibrium.

   b. Show that if the economy has a competitive equilibrium then the equilibrium prices are equal.

   c. Show that if $e(1) = e(2) = (2, 1)$ then the economy has no competitive equilibrium.

   d. Characterize all initial allocations for which the economy has a competitive equilibrium.

6. *Integration of exchange market and housing market.* In the exchange economy $\langle N, (\succeq^i)_{i \in N}, e \rangle$ the initial bundle of each individual differs from the initial bundle of every other individual ($e(i) \neq e(j)$ for all $i \neq j$ in N). Each individual has a monotone, continuous, and convex preference relation. Rather than assuming that each individual can choose any bundle in $\mathbb{R}^2_+$, assume that each individual can choose only one of the bundles held initially by one of the individuals.

   a. Assume that in equilibrium a price is attached to each bundle (not each good). Explain how the housing model of Chapter 9 can be applied to define an equilibrium of the market.

   b. Construct an example of such an economy with four individuals where $e(1) = (2, 0)$, $e(2) = (0, 2)$, $e(3) = (1, 0)$, and $e(4) = (0, 1)$ such that any

competitive equilibrium price function p is not linear in the sense that
for no (p_1, p_2) is it the case that $p(e(i)) = p_1 e_1(i) + p_2 e_2(i)$ for all i.

7. *Economy with differentiable preferences.* Characterize in an Edgeworth box
   (Section 10.9) all the Pareto stable allocations in an exchange economy with
   two individuals in which the sum of the individuals' initial bundles is $(1, 1)$,
   the individuals' preference relations are strictly monotone, convex, and dif-
   ferentiable, and, for each individual, the marginal rate of substitution is less
   than 1 at each bundle (x_1, x_2) for which $x_1 + x_2 > 1$, greater than 1 at each bun-
   dle for which $x_1 + x_2 < 1$, and equal to 1 at each bundle for which $x_1 + x_2 = 1$.

8. *Exchange economy with one indivisible good.* Consider an exchange econ-
   omy $\langle N, (\succsim^i)_{i \in N}, e \rangle$ in which $N = \{1, \ldots, n\}$, where n is odd. Good 1 can be
   consumed only in the amounts 0, 1, or 2 whereas good 2 can be consumed in
   any amount. Assume that each individual i has a preference relation repre-
   sented by the function $t^i x_1 + x_2$, where $t^1 > t^2 > \cdots > t^n > 0$, and initially has
   the bundle $(1, M^i)$, where $M^i > t^1$. Characterize the competitive equilibria of
   this economy.

9. *One individual determines the prices.* Consider an exchange economy with
   two individuals in which individual 2 chooses the price ratio and commits to
   comply with any exchange that individual 1 chooses given that ratio. Individ-
   ual 2 foresees 1's response and chooses the exchange rate so that individual
   1's response is best for her (individual 2). Use an Edgeworth box to show
   graphically the following two results.

   a. The outcome of the procedure might (and typically does) differ from a
      competitive equilibrium, and when it differs it is better for individual 2.

   b. Proposition 10.3 does not hold for the procedure: if the outcome of the
      procedure is the allocation b and the individuals are assigned the bun-
      dles $b(1)$ and $b(2)$ then individual 2 can achieve a bundle better than $b(2)$
      by announcing a price for a trade away from b.

Notes

The modern theory of competitive equilibrium, which has its origins in the work
of Walras (1874), was developed by Kenneth J. Arrow, Gerard Debreu, and Lionel
McKenzie (see for example Arrow and Debreu 1954, Debreu 1959, and McKenzie
1954, 1959). The Edgeworth box (Section 10.9) was introduced by Edgeworth
(1881, 28, 114).