# Models in Microeconomic Theory

**Expanded Second Edition** 

Martin J. Osborne Ariel Rubinstein





## https://www.openbookpublishers.com

© 2020, 2023 (expanded second edition) Martin J. Osborne and Ariel Rubinstein



This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivs license (CC BY-NC-ND 4.0). This license allows you to share, copy, distribute, and transmit the work providing you do not modify the work, you do not use the work for commercial purposes, you attribute the work to the authors, and you provide a link to the license. Attribution should not in any way suggest that the authors endorse you or your use of the work and should include the following information.

Martin J. Osborne and Ariel Rubinstein, *Models in Microeconomic Theory* (Expanded Second Edition), Cambridge, UK: Open Book Publishers, 2023, https://doi.org/10.11647/0BP.0361.

To access detailed and updated information on the license, please visit https://www.openbookpublishers.com/product/1171#copyright.

For further details on CC BY-NC-ND licenses, see https://creativecommons.org/ licenses/by-nc-nd/4.0/. All external links were active at the time of publication unless otherwise stated and have been archived via the Internet Archive Wayback Machine at https://archive.org/web.

Updated digital material and resources associated with this book are available at https: //www.openbookpublishers.com/product/1171#resources.

Every effort has been made to identify and contact copyright holders and any omission or error will be corrected if notification is made to the publisher.

ISBN Paperback: 978-1-80511-121-4 ISBN Hardback: 978-1-80511-122-1 ISBN Digital (PDF): 978-1-80511-123-8 DOI: 10.11647/OBP.0361

This book is available in two versions, one that uses feminine pronouns and one that uses masculine pronouns. This version uses feminine pronouns.

Version: 2023.5.30 (s)

# 12 A market with consumers and producers

This chapter describes two models that extend the model of an exchange economy to economies in which goods are produced. We do not analyze the models in detail, but only prove, for each model, one result regarding the Pareto stability of the equilibrium outcome.

## 12.1 Production economy

## 12.1.1 Introduction

Every day has a morning and an afternoon. All decision-makers face the same price system, which remains the same during the day. Each production unit is controlled by a manager and is owned by consumers. In the morning, each manager chooses a feasible production plan. Here we assume that the manager's objective is to maximize the profit of the production unit, on the assumption that all of the output will be sold at the given prices. After lunch the profit of each production unit is divided among the owners of the unit. Every individual observes the sum of the profits she has received from the production units in which she has an ownership share. In the afternoon she chooses a consumption bundle that is optimal for her in the budget set determined by her income and the price system.

If every consumer is able to purchase a bundle that is optimal for her and no surplus of any good remains, then the producers' and consumers' decisions are in harmony and the prices are consistent with equilibrium. If a surplus or shortage of some good exists (goods remain on the shelves or the shelves are empty and some consumers cannot purchase as much as they desire), then the economy is not in equilibrium, and we expect prices to change.

## 12.1.2 Model

The economy has two goods, 1 and 2, a set *I* of consumers, and a set *J* of producers. Each consumer *i* is characterized by an increasing, continuous, and convex preference relation  $\succeq^i$  on the set of bundles  $\mathbb{R}^2_+$ . Each producer *j* is characterized by a technology, a set  $T(j) \subseteq \mathbb{R}^2_+$  of all the bundles she can produce.

Chapter of *Models in Microeconomic Theory* by Martin J. Osborne and Ariel Rubinstein. Version 2023.5.30 (s). © 2023 Martin J. Osborne and Ariel Rubinstein CC BY-NC-ND 4.0. https://doi.org/10.11647/0BP.0361.12



**Figure 12.1** An example of a technology: a closed, bounded, convex subset of  $\mathbb{R}^2_+$  that includes (0,0) and has the property that for every point  $(z_1, z_2)$  in the set, every point  $(y_1, y_2) \in \mathbb{R}^2_+$  with  $y_1 \leq z_1$  and  $y_2 \leq z_2$  is in the set.

Each producer *j* chooses a member of T(j). Notice that this formalization of the producer's decision is simplistic: she is endowed with production abilities and needs merely to choose a combination of goods to produce. (She incurs no cost.) We assume that each T(j) is a technology, defined as follows and illustrated in Figure 12.1.

#### Definition 12.1: Technology

A *technology T* is a set  $T \subseteq \mathbb{R}^2_+$  that is closed, bounded, and convex, and has the property that if  $(x_1, x_2) \in T(j)$ ,  $y_1 \leq x_1$ , and  $y_2 \leq x_2$  then  $(y_1, y_2) \in T(j)$  (that is, goods can be *freely disposed*).

Each producer j, when choosing the output of her production unit (an element in T(j)) takes as given the price system  $(p_1, p_2)$  prevailing in the market and maximizes the value of this output (the unit's profit). That is, producer j chooses a solution of the problem

$$\max_{x \in T(j)} px$$

where  $p_{x} = p_{1}x_{1} + p_{2}x_{2}$ .

The last element of the model provides a link between the production units' profits and the consumers' budgets. We assume that the profit of each unit is divided among the consumers. Denote by  $\alpha(i, j)$  the fraction of the profit of producer *j* that belongs to consumer *i*. All the profit of each production unit is distributed to consumers, so  $\sum_{i \in I} \alpha(i, j) = 1$  for every *j*. Each consumer chooses a bundle to maximize her preferences given her wealth, which is the total profit she receives. Note that the model takes the ownership shares as given; it does not include the process by which ownership is determined.

## **Definition 12.2: Production economy**

A production economy  $(I, J, (\succeq^i)_{i \in I}, (T(j))_{j \in J}, \alpha)$  consists of

## consumers

a finite set I

### producers

a finite set J

## consumers' preferences

for each consumer  $i \in I$ , a preference relation  $\succeq^i$  over  $\mathbb{R}^2_+$  that is monotone, continuous, and convex

## technologies

for each producer  $j \in J$ , a technology  $T(j) \subseteq \mathbb{R}^2_+$ , the set of bundles that *j* can produce

## ownership shares

for every consumer  $i \in I$  and producer  $j \in J$ , a number  $\alpha(i, j) \in [0, 1]$ with  $\sum_{i \in I} \alpha(i, j) = 1$  for every  $j \in J$ ;  $\alpha(i, j)$  is the fraction of producer *j*'s profit owned by consumer *i*.

A feasible outcome in the economy specifies the bundle chosen by each producer and by each consumer such that the total amount of each good produced is equal to the total amount of each good consumed.

```
Definition 12.3: Consumption-production plan
```

A *consumption-production plan* in the production economy  $(I, J, (\succeq^i)_{i \in I}, (T(j))_{j \in J}, \alpha)$  is a pair (x, y) where  $x = (x(i))_{i \in I}$  is an assignment of bundles to consumers and  $y = (y(j))_{j \in J}$  is an assignment of bundles to producers such that  $y(j) \in T(j)$  for every producer  $j \in J$  and  $\sum_{i \in I} x(i) = \sum_{j \in J} y(j)$ .

A candidate for a competitive equilibrium of a production economy consists of a price system  $p^* = (p_1^*, p_2^*)$ , a consumption decision  $x^*(i)$  for every consumer  $i \in I$ , and a production decision  $y^*(j)$  for every producer  $j \in J$ . A candidate is a competitive equilibrium if the following conditions are satisfied.

- For every consumer *i*, the bundle *x*\*(*i*) is optimal given *p*\* and the income *i* gets from her shares of the producers' profits.
- For every producer *j*, the bundle *y*\*(*j*) maximizes *j*'s profit given *p*\* and her technology *T*(*j*).

• The combination of consumption and production decisions is feasible: it is a consumption-production plan.

Definition 12.4: Competitive equilibrium of production economy

A *competitive equilibrium* of the production economy  $(I, J, (\succeq^i)_{i \in I}, (T(j))_{j \in J}, \alpha)$  is a pair (p, (x, y)) consisting of

- a price system  $p = (p_1, p_2)$  and
- an assignment of bundles to consumers x = (x(i))<sub>i∈I</sub> and an assignment of bundles to producers y = (y(j))<sub>j∈J</sub>

such that

## optimality of consumers' choices

for every consumer  $i \in I$ , the bundle x(i) is maximal according to  $\succeq^i$ in the set  $\{x \in \mathbb{R}^2_+ : px = \sum_{j \in J} \alpha(i, j)\pi(j)\}$ , where  $\pi(j) = py(j)$  for each  $j \in J$  (the profit of producer j)

## optimality of producers' choices

for every producer  $j \in J$ , the bundle y(j) maximizes pz subject to  $z \in T(j)$ 

feasibility

(x, y) is a consumption-production plan.

The notion of Pareto stability can be applied to a production economy: a consumption-production plan is Pareto stable if no consumption-production plan is at least as good for all consumers and better for at least one of them.

Definition 12.5: Pareto stable consumption-production plan

The consumption-production plan (x', y') in the production economy  $\langle I, J, (\succeq^i)_{i \in I}, (T(j))_{j \in J}, \alpha \rangle$  *Pareto dominates* the consumption-production plan (x, y) if  $x'(i) \succeq^i x(i)$  for all  $i \in I$  and  $x'(i) \succ^i x(i)$  for some  $i \in I$ . The consumption-production plan (x, y) is *Pareto stable* if no plan (x', y') Pareto dominates it.

## 12.1.3 Competitive equilibrium

We now show that the consumption-production plan generated by a competitive equilibrium of a production economy is Pareto stable (a counterpart of Proposition 10.4 for an exchange economy).

#### Proposition 12.1: Pareto stability of competitive equilibrium

The consumption-production plan generated by any competitive equilibrium of a production economy is Pareto stable.

#### Proof

Let (p,(x,y)) be a competitive equilibrium of the production economy  $(I, J, (\succeq^i)_{i \in I}, (T(j))_{j \in J}, \alpha)$ . Assume that the consumption-production plan (x', y') Pareto dominates (x, y). The optimality of the producers' choices in the competitive equilibrium implies that  $py(j) \ge py'(j)$  for every  $j \in J$ , so that

$$p\sum_{j\in J} y(j) \ge p\sum_{j\in J} y'(j)$$

Also,  $px'(i) \ge px(i)$  for every consumer  $i \in I$  (if px'(i) < px(i) then given that  $x'(i) \succeq^i x(i)$  and that  $\succeq^i$  is monotone, there is a bundle z with pz < px(i) and  $z \succ^i x(i)$ , contradicting the optimality of x(i)). For the consumer i for whom  $x'(i) \succ^i x(i)$ , we have px'(i) > px(i) (otherwise x(i)is not optimal for i given the price system p). Thus

$$p\sum_{i\in I} x'(i) > p\sum_{i\in I} x(i)$$

But the feasibility requirement of the equilibrium,  $\sum_{i \in I} x(i) = \sum_{j \in J} y(j)$ , so

$$p \sum_{i \in I} x'(i) > p \sum_{i \in I} x(i) = p \sum_{j \in J} y(j) \ge p \sum_{j \in J} y'(j)$$

contradicting  $\sum_{i \in I} x'(i) = \sum_{j \in J} y'(j)$ .

Note that the proof of this result does not use the convexity of the preferences or of the technology. However, without these assumptions a competitive equilibrium may not exist. Consider a production economy with one producer and one consumer, who owns the producer's profit. The consumer's preference relation is convex, and is represented by the function  $\min\{x_1, x_2\}$ . The technology is the set *T* as depicted in Figure 12.2. For any price system, the production bundle that maximizes profit is either *a* or *b* (or both). But for any budget set the consumer's optimal bundle involves equal amounts of the goods. So for no price system does the consumer's optimal bundle coincide with the producer's optimal bundle, as competitive equilibrium requires in this economy.



**Figure 12.2** An economy with production with a single consumer and single production unit in which the technology is not convex.

## 12.2 An economy with capital and labor

## 12.2.1 Introduction

A capitalist uses the labor of a worker to produce a good. Given the wage rate, the capitalist decides how much labor time to buy, and the production process she owns yields a quantity of the good; she uses some of the output to pay the worker and consumes the remainder. The worker decides how long to work, is paid, and consumes her income and any remaining leisure time. In an equilibrium, wages are such that the amount of time the worker wants to work is equal to the quantity of labor the capitalist wants to buy.

## 12.2.2 Model

There are two goods, a consumption good and leisure, and two individuals, a capitalist and a worker. The production process transforms an amount of time into an amount of the consumption good. The production function f describes this process: the output produced by a units of time is f(a). We assume that f is increasing and concave, and satisfies f(0) = 0. (Figure 12.3 shows an example.)

The worker has one unit of time and decides how to divide it between leisure and work. She is characterized by a preference relation on  $\{(l, x) : 0 \le l \le 1, x \ge 0\}$ , where *l* is an amount of leisure and *x* is an amount of the consumption good. We assume that this preference relation is monotone, continuous, and convex.

## **Definition 12.6: Capitalist-worker economy**

A *capitalist-worker economy*  $\langle f, \succeq \rangle$  consists of



Figure 12.3 A production function.

#### capitalist's technology

an increasing concave function  $f : \mathbb{R}_+ \to \mathbb{R}_+$  with f(0) = 0, the *production function* available to the capitalist, which associates with every nonnegative number *a* (an amount of labor) a nonnegative number f(a) (the amount of a consumption good produced).

#### worker's preferences

a monotone, continuous, and convex preference relation  $\succeq$  over  $\mathbb{R}^2_+$  (the worker's preferences over pairs (l, x) consisting of an amount l of leisure and an amount x of the consumption good).

We assume one individual of each type only for simplicity. The model can easily be extended to include multiple capitalists and workers.

Given a wage rate, the producer chooses the amount of labor time to buy. We assume here that she aims to maximize profit. That is, given the wage rate w (measured in units of the consumption good per unit of time) the producer chooses a to maximize f(a) - wa. The worker decides the amount of time l to keep for leisure; she chooses the value of l that generates the pair (l, w(1 - l)) that is best according to her preferences.

## Definition 12.7: Consumption-production plan

A *consumption-production plan* in the capitalist-worker economy  $\langle f, \succeq \rangle$  is a pair ((l, x), (a, z)) consisting of an amount l of leisure for the worker, an amount x of the consumption good assigned to the worker, an employment level a, and an amount z of the consumption good assigned to the capitalist, with a = 1 - l and f(a) = x + z.

The following definition of Pareto stability is appropriate for the model.

## **Definition 12.8: Pareto stability**

A consumption-production plan ((l, x), (a, z)) in the capitalist-worker economy  $\langle f, \succeq \rangle$  is *Pareto stable* if there is no consumption-production plan ((l', x'), (a', z')) for which  $z' \ge z$  and  $(l', x') \succeq (l, x)$ , with at least one strict inequality.

A competitive equilibrium consists of a wage rate  $w^*$ , an employment level  $a^*$ , and a consumption bundle  $(l^*, x^*)$  for the worker such that

- the bundle (*l*\*, *x*\*) is optimal for the consumer and the employment level *a*\* maximizes the capitalist's profit, given the wage rate
- the amount of time the worker wants to devote to production is equal to the amount of labor time the capitalist wants to use (the employment level).

### Definition 12.9: Competitive equilibrium of capitalist-worker economy

A competitive equilibrium of a capitalist-worker economy  $\langle f, \succeq \rangle$  is a pair  $(w^*, ((l^*, x^*), (a^*, z^*)))$  consisting of a positive number  $w^*$  (the wage rate) and a pair of choices, one for the worker,  $(l^*, x^*)$ , and one for the capitalist,  $(a^*, z^*)$ , such that

## optimality of worker's choice

 $(l^*, x^*)$  is maximal with respect to  $\succeq$  in the budget set  $\{(l, x) : 0 \le l \le 1, x = w^*(1-l)\}$  (the worker chooses amounts of leisure and consumption that she likes best given the wage rate)

## optimality of capitalist's choice

 $a^*$  maximizes  $f(a) - w^*a$  (the employment level maximizes the capitalist's profit, given the wage rate) and  $z^* = f(a^*) - w^*a^*$ 

## feasibility

 $((l^*, x^*), (a^*, z^*))$  is a consumption-production plan.

A competitive equilibrium is illustrated in Figure 12.4. Given the wage rate  $w^*$ , the capitalist optimally chooses the employment level  $a^*$ , resulting in the output  $x^* + z^*$ . The worker optimally supplies  $a^*$  units of labor time, earning  $w^*a^*$  and thus facing the budget set indicated. In this set, the optimal bundle for the consumer is  $(l^*, x^*)$ .

## Proposition 12.2: Pareto stability of competitive equilibrium

The consumption-production plan in any competitive equilibrium of a capitalist-worker economy is Pareto stable.

#### Problems





## Proof

Consider a competitive equilibrium  $(w^*,((l^*,x^*),(a^*,z^*)))$ . Let ((l',x'),(a',z')) be a consumption-production plan that Pareto dominates  $((l^*,x^*),(a^*,z^*))$ . Thus  $z' \ge z^*$  and  $(l',x') \succcurlyeq (l,x)$  with at least one strict inequality. By the optimality of  $(l^*,x^*)$  in the set  $\{(l,x): 0 \le l \le 1, x = w^*(1-l)\}$  we have  $x' \ge w^*(1-l')$ . Therefore one of the inequalities  $z' \ge z^*$  and  $x' \ge w^*(1-l')$  must be strict. By the feasibility of the plan, x'+z'=f(a') and a' = (1-l'). Thus  $f(a') - w^*a' = f(a') - w^*(1-l') \ge f(a') - x' = z' \ge z^* = f(a^*) - w^*a^*$  with one of the inequalities strict. Thus  $f(a') - w^*a' > f(a^*) - w^*a^*$ , contradicting the optimality of  $a^*$  for the capitalist.

We close the chapter by emphasizing again that Pareto stability is not a normative notion. The fact that a consumption-production plan is Pareto stable means only that any plan that one of the individuals (the capitalist and the worker) prefers is worse for the other individual. A competitive equilibrium may be just or unjust; a regulation like a minimum wage may lead to a consumptionproduction plan that is not Pareto stable but is fairer.

### Problems

1. *Comparative advantage and specialization*. Consider an economy with two goods and a set  $N = \{1, ..., n\}$  of individuals. Each individual is both a consumer and a producer. Individual *i* chooses a bundle from the set  $T(i) = \{(y_1, y_2) : t^i y_1 + y_2 \le c^i\}$ , where  $c^i$  and  $t^i$  are positive constants, with  $t^1 < t^2 < \cdots < t^n$ . Each individual can trade the bundle she produces for another bundle at the market prices.

- *a*. Given a price system *p*, define a *p*-production-consumption plan for individual *i* to be a pair (x(i), y(i)) such that  $y(i) \in T(i)$  and px(i) = py(i). Define an appropriate concept of competitive equilibrium.
- *b*. Show that given the price system  $(p_1, p_2)$ , every individual for whom  $t^i < p_1/p_2$  produces only good 1, every individual for whom  $t^i > p_1/p_2$  produces only good 2, and every individual for whom  $t^i = p_1/p_2$  are indifferent between all *p*-production-consumption plans.
- *c*. Show that if all individuals have the same preference relation, represented by the utility function  $tx_1 + x_2$ , then the economy has a competitive equilibrium in which each individual consumes the bundle that she produces.
- *d*. Assume that n = 2 and each individual has preferences represented by the utility function  $\min\{x_1, x_2\}$ . Give an example of an economy in which  $t^1 < t^2$  (individual 1 has a *comparative advantage* in producing good 1) with a competitive equilibrium in which individual 1 produces both goods.
- 2. *Capitalist-worker economy with output-maximizing capitalist.* Assume that in a capitalist-worker economy the capitalist maximizes output subject to the constraint that profit is nonnegative (see Section 6.2). Illustrate in a diagram like Figure 12.4 a competitive equilibrium of the economy. Is an equilibrium outcome necessarily Pareto stable?
- 3. *Technological improvement in capitalist-worker economy*. Show by examples that a technological improvement in a capitalist-worker economy (in which the capitalist maximizes profit) may change the competitive equilibrium so that the capitalist is worse off or the worker is worse off.
- 4. *Production chain.* Consider an economy with two producers. Producer 1 makes the good X using her own labor time; t units of time generate the output f(t). Producer 2 makes good Y using X as an input; her production function is g. Both f and g are strictly concave, increasing, and differentiable. Producer 1 has a differentiable, monotone, and convex preference relation over pairs consisting of amounts of Y and leisure. Producer 2 chooses the amount of X to maximize her profit. Each producer is the sole owner of her technology.

A candidate for a competitive equilibrium consists of (*i*) a price  $p^*$  of X in terms of Y, (*ii*) the amount of time  $t^*$  that producer 1 devotes to making X, and (*iii*) the quantity  $x^*$  of X that producer 2 uses. A candidate ( $p^*, t^*, x^*$ ) is a competitive equilibrium if (*i*) producer 1's decision maximizes her preference

relation given  $p^*$ , (*ii*) producer 2's decision maximizes her profit, and (*iii*) the supply of *X* by producer 1 is equal to the demand for *X* by producer 2.

Show that the outcome of a competitive equilibrium is Pareto stable.

5. *Pollution*. In an economy in which one individual's action has a direct effect on another individual, a competitive equilibrium may not be Pareto stable. To demonstrate this point, consider an economy with two goods, *N* producers, and *N* consumers. Each producer has the production technology  $T = \{(y_1, y_2) : 2y_1 + y_2 = 2\}$  (and incurs no cost), and maximizes her profit. The producers' profits are divided equally among all consumers. Consumption of good 2 produces pollution. The pollution index is 1.5 times the average consumption of good 2. Each consumer has the utility function  $x_1 + x_2 - z$ , where *z* is the pollution index. When choosing a bundle a consumer takes the pollution index as given. (This assumption seems reasonable when *N* is large.) Define an appropriate notion of symmetric competitive equilibrium in which all consumers choose the same bundle and all producers choose the same member of *T*. Show that any symmetric equilibrium outcome is not Pareto stable.