Models in Microeconomic Theory

Expanded Second Edition

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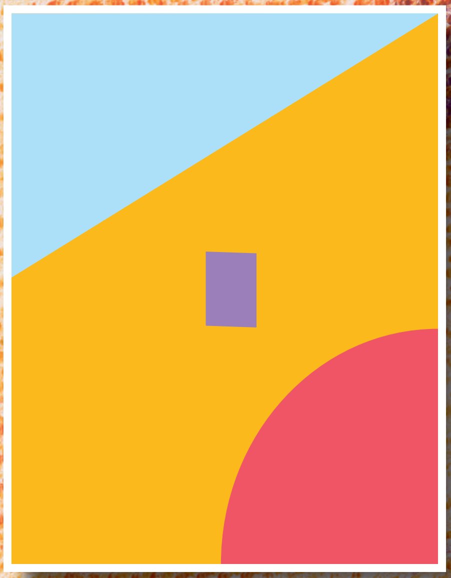

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15 Strategic games

The model of a strategic game is central to game theory. In a strategic game, each individual chooses an action from a given set and is affected not only by this action but also by the other individuals' actions. We study mainly the notion of Nash equilibrium, according to which a profile of actions is stable if no individual wants to deviate from her action given the other individuals' actions.

15.1 Strategic games and Nash equilibrium

A strategic game consists of a set of players, each of whom is characterized by the set of actions available to her and a preference relation over action profiles (lists of actions, one for each player). Each player chooses one of her available actions, so that an outcome of the game is an action profile. We often work with utility functions that represent the players' preference relations, rather than explicitly with preferences, and refer to the utility functions as payoff functions.

Definition 15.1: Strategic game

A *strategic game* $\langle N, (A^i)_{i\in N}, (\succsim^i)_{i\in N}\rangle$ consists of

players
a set $N = \{1, \ldots, n\}$

actions
for each player $i \in N$, a set A^i of actions

preferences
for each player $i \in N$, a preference relation $\succsim^i$ over the set $A = \times_{i\in N} A^i$ of *action profiles*.

A function $u^i : A \to \mathbb{R}$ that represents $\succsim^i$ is a *payoff function* for player i.

This model differs from the models discussed in Part II in two main ways. First, in a strategic game the set of alternatives of each player is fixed, whereas in the market models the set of alternatives available to an individual is determined by the equilibrium. Second, in the market models an individual's preferences are defined over her own choices, whereas in a strategic game a player's preferences

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are defined over the set of action profiles, so that they take into account the effect of other players' actions on the player.

The main solution concept we study is Nash equilibrium. A Nash equilibrium is an action profile with the property that no deviation by any player leads to an action profile that the player prefers. That is, every player's action in a Nash equilibrium is best for her given the other players' actions.

Definition 15.2: Nash equilibrium of strategic game

In a strategic game $\langle N, (A^i)_{i\in N}, (\succsim^i)_{i\in N}\rangle$, an action profile $a = (a^i) \in A$ is a *Nash equilibrium* if for every player $i \in N$ we have

$$(a^i, a^{-i}) \succsim^i (x^i, a^{-i}) \text{ for all } x^i \in A^i$$

where (x^i, a^{-i}) denotes the action profile that differs from a only in that the action of individual i is x^i rather than a^i.

Like the other equilibrium concepts we discuss, Nash equilibrium is static: we do not consider either a dynamic process or a reasoning process that might lead each player to choose her Nash equilibrium action. Note also that the notion of Nash equilibrium does not consider the instability that might arise if groups of players act together. It simply identifies outcomes that are stable against deviations by individuals, without specifying how these outcomes are attained.

We can express the condition for a Nash equilibrium differently using the notion of a best response.

Definition 15.3: Best response

In a strategic game $\langle N, (A^i)_{i\in N}, (\succsim^i)_{i\in N}\rangle$, the action $a^i \in A^i$ of player i is a *best response* to the list a^{-i} of the other players' actions if

$$(a^i, a^{-i}) \succsim^i (x^i, a^{-i}) \text{ for all } x^i \in A^i.$$

Denote by $BR(a^{-i})$ the set of player i's best responses to a^{-i}. Then an action profile a is a Nash equilibrium if and only if $a^i \in BR(a^{-i})$ for each player i.

15.2 Basic examples

Example 15.1: Traveler's dilemma

Each of two people chooses a number of dollars between \$180 and \$300. Each person receives the lower of the two amounts chosen. In addition, if the amounts chosen differ, \$5 is transferred from the person who chose the larger amount to the person who chose the smaller one. (If the amounts chosen are the same, no transfer is made.)

The name traveler's dilemma comes from a story used to add color (a part of the charm of game theory). Each of two travelers takes a suitcase containing an identical object on a flight. The value of the object is known to be between \$180 and \$300. The suitcases are lost and the airline has to compensate the travelers. The airline asks each traveler to name an integer between 180 and 300. Each traveler gets (in dollars) the smaller of the numbers chosen, and, if the numbers differ, in addition \$5 is transferred from the traveler who names the larger number to the one who names the smaller number.

A strategic game that models this situation has $N = \{1,2\}$, $A^i = \{180, 181, \ldots, 300\}$ for $i = 1, 2$, and

$$u^i(a^1, a^2) = \begin{cases} a^i + 5 & \text{if } a^i < a^j \\ a^i & \text{if } a^i = a^j \\ a^j - 5 & \text{if } a^i > a^j, \end{cases}$$

where j is the player other than i.

Claim *The only Nash equilibrium of the traveler's dilemma is* $(180, 180)$.

Proof. First note that $(180, 180)$ is indeed a Nash equilibrium. If a player increases the number she names, her payoff falls by 5.

No other pair (a^1, a^2) is an equilibrium. Without loss of generality, assume $a^1 \geq a^2$. If $a^1 > a^2$, then a deviation of player 1 to a^2 increases her payoff from $a^2 - 5$ to a^2. If $a^1 = a^2 \neq 180$, then a deviation of player i to $a^i - 1$ increases her payoff from a^i to $a^i + 4$. ◁

When people are asked to play the game (without the suitcase interpretation), most say they would choose a number different from 180. For example, among 21,000 students of courses in game theory around the world who have responded at `https://arielrubinstein.org/gt`, only 22% have chosen 180. The most popular choice is 300 (43%). About 8% chose 299 and 7% chose a number in the range 295–298. The action 298 is the best action given the distribution of the participants' choices.

One possible explanation for the difference between these results and Nash equilibrium is that the participants' preferences are not those specified in the game. Most people care not only about the dollar amount they receive. Some perceive 300 to be the socially desirable action especially if they anticipate that most other people would choose 300. Many people dislike gaining a few dollars at the expense of another person, especially if they believe the other person is not trying to game the system. Thus, for example, player 1 may prefer the outcome (300, 300) to (299, 300), even though the latter involves a higher monetary reward. In this case, the experimental results conflict less with Nash equilibrium as (300, 300) is an equilibrium in the game with these modified preferences.

The next few examples are two-player games with a small number of alternatives for each player. Such a game may conveniently be presented in a table with one row for each action of player 1, one column for each action of player 2, and two numbers in each cell, that are payoffs representing the players' preferences. For example, the following table represents a game in which player 1's actions are T and B, and player 2's are L and R. Each cell corresponds to an action profile. For example, the top left cell corresponds to (T, L). The preferences of player 1 over the set of action profiles are represented by the numbers at the left of each cell and those of player 2 are represented by the numbers at the right of each cell. Thus, for example, the worst action profile for player 1 is (B, R) and the best action profile for player 2 is (B, L).

	L	R
T	$5, 0$	$-1, 1$
B	$3, 7$	$-2, 0$

Example 15.2: Prisoner's dilemma

The Prisoner's dilemma is the most well-known strategic game. The story behind it involves two suspects in a robbery who are caught conducting a lesser crime. The police have evidence regarding only the lesser crime. If both suspects admit to the robbery, each is sentenced to six years in jail. If one of them admits to the robbery and implicates the other, who does not admit to it, then the former is set free and the latter is sentenced to seven years in jail. If neither admits to the robbery then each is sentenced to one year in jail. Each person aims to maximize the number of free years within the next seven years.

The structure of the incentives in this story is shared by many other situations. The essential elements are that each of two individuals has to choose between two courses of action, C (like not admitting) and D

(like admitting), each individual prefers D to C regardless of the other individual's action, and both individuals prefer (C,C) to (D,D).

We can model the situation as a strategic game in which $N = \{1,2\}$, $A^i = \{C,D\}$ for $i = 1,2$, and the players' preferences are represented by the payoffs in the following table.

	C	D
C	6,6	0,7
D	7,0	1,1

Each player's optimal action is D, independent of the other player's action. Thus (D,D) is the only Nash equilibrium of the game.

The action profile (D,D) is not Pareto stable: both players prefer (C,C). This fact sometimes leads people to use the game to argue that rational behavior by all players may lead to an outcome that is socially undesirable.

Note that in the situation the game is intended to model, some people, at least, would probably not have the preferences we have assumed: the guilt from choosing D when the other person chooses C would lead them to prefer (C,C) to the action profile in which they choose D and the other person chooses C. In the game in which each player has such modified preferences, (C,C) is a Nash equilibrium.

The previous two strategic games, the traveler's dilemma and the prisoner's dilemma, are symmetric: the set of actions of each player is the same and the payoff of player 1 for any action pair (a^1,a^2) is the same as the payoff of player 2 for the action pair (a^2,a^1).

Definition 15.4: Symmetric two-player game

A two-player strategic game $\langle\{1,2\},(A^i)_{i\in\{1,2\}},(\succsim^i)_{i\in\{1,2\}}\rangle$ is *symmetric* if $A^1 = A^2$ and $(a^1,a^2)\succsim^1(b^1,b^2)$ if and only if $(a^2,a^1)\succsim^2(b^2,b^1)$.

In other words, if u^1 represents $\succsim^1$ then the function u^2 defined by $u^2(a^1,a^2) = u^1(a^2,a^1)$ represents $\succsim^2$. In a symmetric game, a player's preferences can be described by using only the terms "the player" and "the other player", without referring to the player's name.

Example 15.3: Where to meet? (Bach or Stravinsky)

Two people can meet at one of two locations, B (perhaps a concert of music by Bach) or S (perhaps a concert of music by Stravinsky). One person prefers to meet at B and the other prefers to meet at S. Each person prefers to meet somewhere than not to meet at all and is indifferent between the outcomes in which she alone shows up at one of the locations.

This situation is modeled by the following game.

	B	S
B	2,1	0,0
S	0,0	1,2

The game has two Nash equilibria, (B,B) and (S,S). The first equilibrium can be thought of as representing the convention that player 2 yields to player 1, while the second equilibrium represents the convention that player 1 yields to player 2. These interpretations are particularly attractive if the people who engage in the game differ systematically. For example, if player 1 is older than player 2, then the first equilibrium can be interpreted as a norm that the younger player yields to the older one.

Note that the situation can be modeled alternatively as a symmetric game where each player has the two actions F (favorite) and N, as follows.

	F	N
F	0,0	2,1
N	1,2	0,0

Although this game is symmetric, its two Nash equilibria, (N,F) and (F,N), are not symmetric.

Example 15.4: Odds or evens (matching pennies)

In a two-person game played by children, each player presents between 1 and 5 fingers. One player, say player 1, wins if the sum is odd and the other player, 2, wins if the sum is even. Each player prefers to win than to lose.

In one strategic game that models this situation, each player has five actions.

	1	2	3	4	5
1	0,1	1,0	0,1	1,0	0,1
2	1,0	0,1	1,0	0,1	1,0
3	0,1	1,0	0,1	1,0	0,1
4	1,0	0,1	1,0	0,1	1,0
5	0,1	1,0	0,1	1,0	0,1

Obviously, what matters is only whether a player chooses an odd or even number of fingers. So in another strategic game that models the situation, each player's actions are *odd* and *even*.

	odd	*even*
odd	0,1	1,0
even	1,0	0,1

Unlike the previous two examples, these games are *strictly competitive*: an outcome is better for player 1 if and only if it is worse for player 2. Neither game has a Nash equilibrium. That makes sense: no deterministic stable mode of behavior is to be expected given that the game is used to make random choices. We return to the game in Section 15.7, when discussing a notion of equilibrium that involves randomization.

Note that if each of two players has to choose a side of a coin, *Head* and *Tail*, and player 1 prefers to mismatch player 2's choice whereas player 2 prefers to match 1's choice, we get the same payoffs. For this reason, such an interaction is known also as *matching pennies*.

Example 15.5: Coordination game

Two people can meet at one of three stadium gates, Yellow, Blue, or Green. They want to meet and do not care where. This situation is modeled by the following strategic game.

	Y	B	G
Y	1,1	0,0	0,0
B	0,0	1,1	0,0
G	0,0	0,0	1,1

Each of the three action pairs in which both players choose the same action is a Nash equilibrium. An equilibrium, for example (Yellow, Yellow), makes sense if the Yellow gate is a salient meeting place.

15.3 Economic examples

We start with two examples of auctions. In a *sealed-bid auction*, n players bid for an indivisible object. Player i's monetary valuation of the object is $v^i > 0$, $i = 1, \ldots, n$. Assume for simplicity that no two players have the same valuation, so that without loss of generality $v^1 > v^2 > \cdots > v^n$. Each player's bid is a nonnegative number, and the object is given to the player whose bid is highest; in case of a tie, the object is given to the player with the lowest index among those who submit the highest bid. That is, the winner $W(b^1, \ldots, b^n)$ is the smallest i such that

$b^i \geq b^j$ for $j = 1, \ldots, n$. We assume that each player cares only about whether she wins the object and how much she pays and, for example, does not regret that she did not bid slightly more if doing so would have caused her to win. The auctions we study differ in the rule determining the amount the players pay. If player i wins and pays p then her payoff is $v^i - p$ and if she does not win and pays p then her payoff is $-p$.

Example 15.6: First-price auction

A first-price auction is a sealed-bid auction in which the player who wins the object (the one with the lowest index among the players whose bids are highest) pays her bid and the others pay nothing, so that player i's payoff function is

$$u^i(b^1, \ldots, b^n) = \begin{cases} v^i - b^i & \text{if } W(b^1, \ldots, b^n) = i \\ 0 & \text{otherwise.} \end{cases}$$

This game has many Nash equilibria. Here are some of them.

- $b^1 = v^2$ and $b^i = v^i$ for all other i. Player 1's payoff is $v^1 - v^2$. She cannot increase her payoff: if she lowers her bid then she is no longer the winner, so that her payoff falls to 0. Any other player can obtain the object only if she bids more than v^2, which causes her payoff to be negative.
- $b^1 = b^2 = v^2$ and $b^i = 0$ for all other i.
- $b^i = p$ for all i, where $v^2 \leq p \leq v^1$.

Claim *In all Nash equilibria of a first-price auction, player 1 gets the object and pays a price in* $[v^2, v^1]$.

Proof. Let $(b^1, \ldots, b^n)$ be a Nash equilibrium. Suppose the winner is $i \neq 1$. We need $b^i \leq v^i$ (otherwise player i's payoff is negative, and she can increase it to 0 by bidding 0). Thus player 1 can deviate to a bid between v^2 and v^1, thereby winning the object and getting a positive payoff.

Now suppose that the winner is 1. We have $b^1 \leq v^1$ (as before). If $b^1 < v^2$ then player 2 can raise her bid to a number between b^1 and v^2 and get a positive payoff. Thus $b^1 \in [v^2, v^1]$. ◁

In fact, $(b^1, \ldots, b^n)$ is a Nash equilibrium of the game if and only if $b^1 \in [v^2, v^1]$, $b^i \leq b^1$ for all $i \neq 1$, and $\max_{i \neq 1} b^i = b^1$.

Example 15.7: Second-price auction

A second-price auction is a sealed-bid auction in which the player who wins the object pays the highest of the *other* bids and the other players pay nothing, so that player i's payoff function is

$$u^i(b^1,\dots,b^n) = \begin{cases} v^i - \max_{j\neq i}\{b^j\} & \text{if } W(b^1,\dots,b^n) = i \\ 0 & \text{otherwise.} \end{cases}$$

To get some intuition about the Nash equilibria of this game, suppose first that $n = 2$, $v^1 = 10$, and $v^2 = 5$. In this case the Nash equilibria of the game include $(7,7)$, $(8,2)$, $(3,12)$, and $(10,5)$.

We now show that the auction has a wide range of Nash equilibria.

Claim *For every player i and every price $p \leq v^i$ a second-price auction has a Nash equilibrium in which i obtains the object and pays p.*

Proof. Consider the action profile in which player i bids $b^i > v^1$, some other player j bids p, and every other player bids 0. Player i wins and her payoff is $v^i - p$. If she changes her bid, her payoff either remains the same or becomes 0. The payoff of every other player is 0, and remains 0 unless she increases her bid and becomes the winner, in which case her bid must be at least $b^i > v^1$, so that her payoff is negative. ◁

The result shows, in particular, that the auction has an equilibrium in which each player bids her valuation. This equilibrium is attractive because it has the special property that regardless of the other players' bids, i's action, to bid her valuation v^i, is at least as good for her as any other action: $u^i(v^i, b^{-i}) \geq u^i(b^i, b^{-i})$ for all b^{-i} and b^i. We return to this property in Chapter 17 (see Problem 2). Although the equilibrium is attractive, in experiments a majority of subjects do not bid their valuations (see for example Kagel and Levin 1993).

Example 15.8: Location game

The inhabitants of a town are distributed uniformly along the main street, modeled as the interval $[0,1]$. Two sellers choose locations in the interval. Each inhabitant buys a unit of a good from the seller whose location is closer to her own location. Thus if the sellers' locations are a^1 and a^2 with $a^1 < a^2$ then all inhabitants with locations less than $\frac{1}{2}(a^1 + a^2)$ patronize seller 1 and all inhabitants with locations greater than $\frac{1}{2}(a^1 + a^2)$ patronize seller 2; the fraction of inhabitants with location exactly $\frac{1}{2}(a^1+a^2)$ is zero, so we can ignore it. Each seller wants to sell to the largest proportion of inhabitants possible.

We can model this situation as a strategic game in which $N=\{1,2\}$ and, for $i=1$, 2, $A^i=[0,1]$ and

$$u^i(a^1,a^2)=\begin{cases}\frac{1}{2}(a^1+a^2) & \text{if } a^i<a^j\\ \frac{1}{2} & \text{if } a^i=a^j\\ 1-\frac{1}{2}(a^1+a^2) & \text{if } a^i>a^j.\end{cases}$$

Note that the game is strictly competitive.

Claim *The only Nash equilibrium of the location game is* $(\frac{1}{2},\frac{1}{2})$.

Proof. The action pair $(\frac{1}{2},\frac{1}{2})$ is a Nash equilibrium: any deviation from $\frac{1}{2}$ by a seller reduces the fraction of inhabitants who patronize the seller.

The game has no other equilibria. If the sellers choose different locations then for each of them a deviation towards the other improves her market share. If the sellers choose the same location, different from $\frac{1}{2}$, then a deviation by a seller to $\frac{1}{2}$ increases the proportion of inhabitants who patronize her. ◁

Notice that a player who chooses the location $\frac{1}{2}$ guarantees that her payoff is at least $\frac{1}{2}$, and a player cannot guarantee to herself more than $\frac{1}{2}$. Such an action, which guarantees a certain payoff and has the property that no other action guarantees a higher payoff, is called a *maxmin action*.

Comment The variant of the game with three players has no Nash equilibrium, by the following argument.

For an action profile in which all players' locations are the same, either a move slightly to the right or a move slightly to the left, or possibly both, increase a player's payoff to more than $\frac{1}{3}$.

For any other action profile, there is a player who is the only one choosing her location and the locations of both other players are either to the left or to the right of her location. Such a player can increase her payoff by moving closer to the other players' locations.

Example 15.9: Effort game

Two players are involved in a joint project. Each player chooses an effort level in $A^i=[0,\infty)$. A player who chooses the level e bears the quadratic

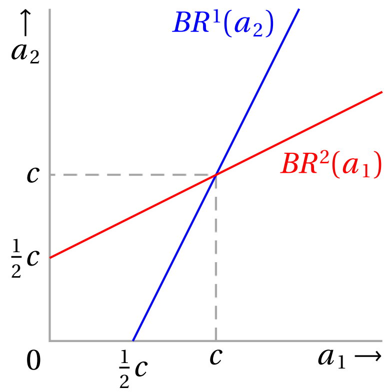


Figure 15.1 The players' best response functions for the effort game in Example 15.9. The game has a unique Nash equilibrium, (c,c).

cost e^2. The project yields player i the amount $a^i(c+a^j)$, where j is the other player and c is a positive constant. Player i's payoff function is given by $u^i(a^i,a^j)=a^i(c+a^j)-(a^i)^2$.

A simple calculation shows that each player i's unique best response to a^j is $\frac{1}{2}(c+a^j)$; the best response functions are shown in Figure 15.1. The equations $a^1 = BR^1(a^2) = \frac{1}{2}(c+a^2)$ and $a^2 = BR^2(a^1) = \frac{1}{2}(c+a^1)$ have a unique solution, $(a^1,a^2)=(c,c)$ (the intersection of the lines in the figure), which is thus the unique Nash equilibrium of the game.

Example 15.10: Quantity-setting oligopoly (Cournot)

Two producers of a good compete in a market. Each of them chooses the quantity of the good to produce. When the total amount they produce is Q, the price in the market is $1-Q$ if $Q \le 1$ and 0 otherwise. Each producer incurs the cost cq when she produces q units, where $c \in (0,1)$, and aims to maximize her profit.

This situation is modeled by a strategic game in which $N = \{1,2\}$ and for $i = 1, 2$, $A^i = [0,1]$ and

$$u^i(q^i,q^j) = \begin{cases} (1-q^i-q^j-c)q^i & \text{if } q^i+q^j \le 1 \\ -cq^i & \text{if } q^i+q^j > 1 \end{cases}$$

(where j is the other player).

To find the Nash equilibria of the game, we find the best response function of each player i. If $q^j > 1-c$ then for every output of player i the price is less than c, so that i's profit is negative; in this case her optimal output is 0. Otherwise her optimal output is $(1-q^j-c)/2$.

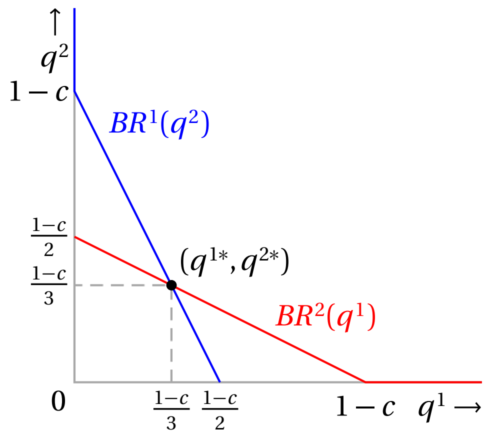


Figure 15.2 The best response functions in a quantity-setting duopoly game in which the inverse demand function is $P = 1 - Q$ and the cost function of each firm is cq. The unique Nash equilibrium is $(q^{1*}, q^{2*}) = (\frac{1}{3}(1-c), \frac{1}{3}(1-c))$.

The best response functions are shown in Figure 15.2. They intersect at $(q^1, q^2) = (\frac{1}{3}(1-c), \frac{1}{3}(1-c))$, which is thus the only Nash equilibrium of the game.

More generally, with n producers the payoff function of player i is

$$u^i(q^1, \ldots, q^n) = \begin{cases} (1 - q^i - \sum_{j \neq i} q^j - c)q^i & \text{if } \sum_{j=1}^n q^j \leq 1 \\ -cq^i & \text{if } \sum_{j=1}^n q^j > 1. \end{cases}$$

Thus player i's best response function is

$$BR^i(q^{-i}) = \max\{0, \tfrac{1}{2}(1 - c - \textstyle\sum_{j \neq i} q^j)\}.$$

In equilibrium, $2q^i = 1 - c - \sum_{j \neq i} q^j$ for each i, so that $q^i = 1 - c - \sum_{i=1}^n q^j$ for each i. Therefore q^i is the same for all i, and is thus equal to $(1-c)/(n+1)$. The price in this equilibrium is $1 - (1-c)n/(n+1)$. As $n \to \infty$ this price converges to c and each producer's profit converges to 0.

Example 15.11: Price-setting duopoly (Bertrand)

As in the previous example, two profit-maximizing producers of a good compete in a market with a mass of consumers of size 1. But now we assume that each of them chooses a price (rather than a quantity). Each consumer buys one unit of the good from the producer whose price is lower if this price is at most 1 and nothing otherwise; if the prices are the same, the consumers are split equally between the producers. Each firm produces the amount demanded from it and in doing so incurs the cost $c \in [0,1)$ per unit. Thus, if producer i's price p^i is lower than producer j's price p^j, producer i's payoff is $p^i - c$ if $p^i \leq 1$ and 0 if $p^i > 1$, and producer j's payoff is 0; if the prices are the same, equal to p, each producer's payoff is $\frac{1}{2}(p-c)$ if $p \leq 1$ and 0 if $p > 1$.

In the strategic game that models this situation, for some actions of one producer the other producer has no best response: if one producer's price is between c and 1, the other producer has no optimal action. (A price slightly lower than the other producer's price is a good response in this case, but given that price is modeled as a continuous variable, no price is optimal.) Nevertheless, the game has a unique Nash equilibrium.

Claim *The only Nash equilibrium of a price-setting duopoly is* (c,c).

Proof. The action pair (c,c) is a Nash equilibrium: if a producer increases her price her profit remains 0, and if she reduces her price her profit becomes negative.

We now argue that (c,c) is the only Nash equilibrium. Suppose that (p^1,p^2) is a Nash equilibrium.

We have $\min\{p^1,p^2\} \geq c$, since otherwise a producer who charges $\min\{p^1,p^2\}$ makes a loss, which she can avoid by raising her price to c. Also $\min\{p^1,p^2\} \leq 1$, since otherwise each producer's payoff is 0 and either producer can increase her payoff by reducing her price to 1.

Also, $p^1 = p^2$, because if $c \leq p^i < p^j$ (and $p^i \leq 1$) then i can increase her payoff by raising her price to any value less than $\min\{1,p^j\}$.

Finally, if $c < p^1 = p^2 \leq 1$ then each player's payoff is positive and either player can reduce her price slightly and almost double her payoff. ◁

15.4 Existence of Nash equilibrium

As we have seen (Example 15.4) some strategic games do not have a Nash equilibrium. We now present two results on the existence of Nash equilibrium in certain families of games. More general results use mathematical tools above the level of this book.

15.4.1 Symmetric games

Our first result is for a family of two-player symmetric games in which each player's set of actions is a closed and bounded interval and her best response function is continuous.

Proposition 15.1: Existence of Nash equilibrium in symmetric game

Let $G = \langle \{1,2\}, (A^i)_{i\in\{1,2\}}, (\succsim^i)_{i\in\{1,2\}} \rangle$ be a two-player symmetric game in which $A^1 = A^2 = I \subset \mathbb{R}$ is a closed and bounded interval for $i = 1, 2$, and each player has a unique best response to every action of the other player, which is a continuous function of the other player's action. Then there exists $x \in I$ such that (x,x) is a Nash equilibrium of G.

Proof

Let $I = [l,r]$. Under the assumptions of the result, the function $g(x) = BR^1(x) - x$, where BR^1 is player 1's best response function, is a continuous function from $[l,r]$ to $\mathbb{R}$ with $g(l) \geq 0$ and $g(r) \leq 0$. Thus by the intermediate value theorem there is a number x^* such that $g(x^*) = 0$, or $BR^1(x^*) = x^*$. Given that the game is symmetric, also $BR^2(x^*) = x^*$, so that (x^*, x^*) is a Nash equilibrium.

An example in which each player's best response is increasing in the other player's action is shown in Figure 15.3a. In this example the game has more than one equilibrium. Such a game does not have any asymmetric equilibria: if (x,y) with $x > y$ were an equilibrium then we would have $BR^1(x) = y$ and $BR^2(y) = x$ and thus also $BR^1(y) = x$, contradicting the assumption that the function BR^1 is increasing.

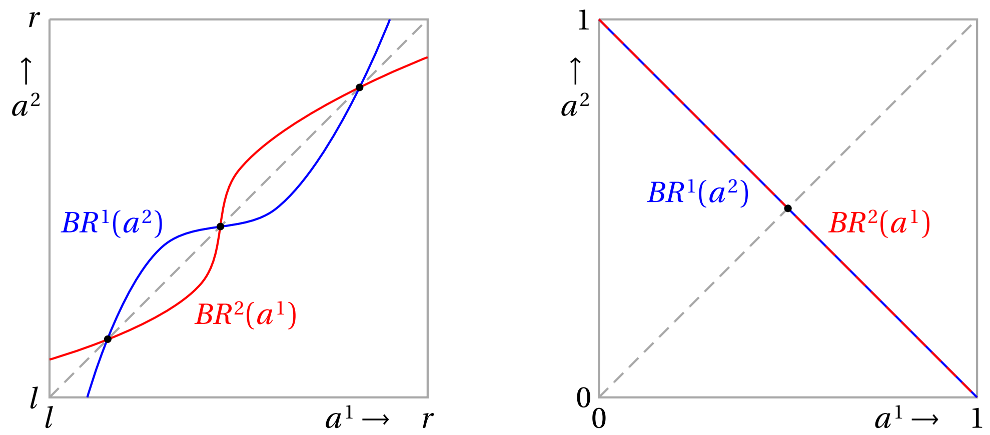
If each player's best response is decreasing in the other player's action then in addition to symmetric equilibria, the game may have asymmetric equilibria, as in the following example.

Example 15.12

Consider the two-player symmetric game where $N = \{1,2\}$, $A^i = [0,1]$, and $u^i(a^1,a^2) = -|1-a^1-a^2|$ for $i = 1, 2$. The best response function of player i is given by $BR^i(a^j) = 1 - a^j$, which is continuous. Every action pair $(x, 1-x)$ for $x \in [0,1]$ is an equilibrium; only $(\frac{1}{2}, \frac{1}{2})$ is symmetric. The best response functions, which coincide, are shown in Figure 15.3b.

15.4.2 Supermodular finite games

The next result concerns games in which the players' best response functions are nondecreasing; such games are called *supermodular*.



(a) The players' best response functions in a symmetric two-player game. The three small black disks indicate the Nash equilibria.

(b) The players' best response functions in the game in Example 15.12. Every pair $(x, 1-x)$ is a Nash equilibrium. The small black disk indicates the symmetric equilibrium.

Figure 15.3

Proposition 15.2: Existence of equilibrium in finite supermodular two-player game

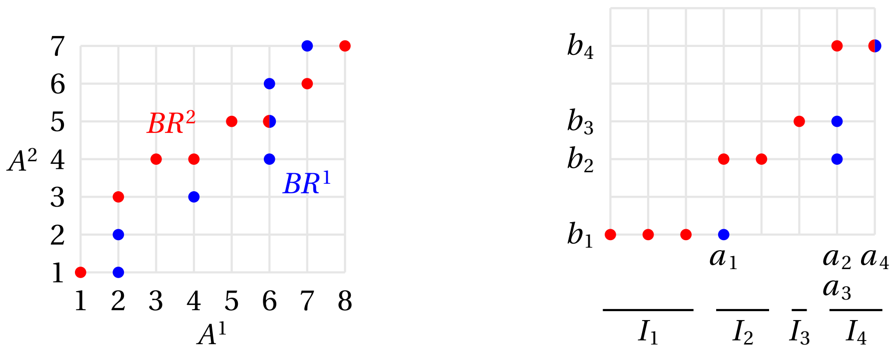
Consider a two-player strategic game $\langle\{1,2\},(A^i)_{i\in\{1,2\}},(\succsim^i)_{i\in\{1,2\}}\rangle$ in which $A^1 = \{1,\ldots,K\}$, $A^2 = \{1,\ldots,L\}$, and all payoffs of each player are distinct, so that each player has a unique best response to each action of the other player. If each best response function is nondecreasing then the game has a Nash equilibrium.

The result assumes that all payoffs of each player are distinct only for simplicity; it remains true without this assumption.

Figure 15.4a shows an example of best response functions satisfying the conditions in the result, with $K = 8$ and $L = 7$. The function BR^1 is indicated by the blue disks and the function BR^2 is indicated by the red disks. The action pair $(6,5)$, colored both blue and red, is a Nash equilibrium.

Proof

Partition the set A^1 into intervals $I_1,\ldots,I_M$ such that for all actions in any given interval I_m the best response of player 2 is the same, equal to b_m ($BR^2(x) \equiv b_m$ for all $x \in I_m$) and $b_m < b_{m'}$ for $m < m'$. (See Figure 15.4b for an illustration.) For each $m = 1,\ldots,M$, let $BR^1(b_m) = a_m$. If for some value of m we have $a_m \in I_m$ then (a_m, b_m) is a Nash equilibrium. Otherwise,



(a) Best response functions in a game satisfying the conditions of Proposition 15.2. The action pair $(6,5)$ is a Nash equilibrium of the game.

(b) An illustration of the proof of Proposition 15.2.

Figure 15.4

denote by $I_{m(i)}$ the interval to which a_i belongs. The fact that BR^2 is nondecreasing implies that $m(i+1) \geq m(i)$ for all i. Now, $m(1) > 1$ (otherwise $(1,1)$ is an equilibrium) and thus $m(2) \geq m(1) \geq 2$, which implies $m(2) > 2$ (otherwise $(2,2)$ is an equilibrium). Continuing the argument we get $m(M) > M$, contradicting $m(M) \leq M$.

15.5 Strictly competitive games

A strategic game is strictly competitive if it has two players and the interests of the players are completely opposed.

Definition 15.5: Strictly competitive game

A two-player strategic game $\langle \{1,2\}, (A^i)_{i\in\{1,2\}}, (\succsim^i)_{i\in\{1,2\}} \rangle$ is *strictly competitive* if for any action pairs a and b,

$$a \succsim^1 b \text{ if and only if } b \succsim^2 a.$$

Strictly competitive games are often called zero-sum games. The reason is that if the function u^1 represents $\succsim^1$ then $\succsim^2$ can be represented by the function u^2 defined by $u^2(a) = -u^1(a)$ for each action pair a, in which case the sum of the players' payoffs is zero for every action pair.

Most economic situations have elements of both conflicting and common interests, and thus cannot be modeled as strictly competitive games. The family of strictly competitive games fits situations for which the central ingredient of the interaction is conflictual. For example, the game of chess is strictly competitive (assuming that each player prefers to win than to tie than to lose). The competition between two politicians for votes may also be modeled as a strictly competitive game.

Consider the following (pessimistic) reasoning by a player: "Whatever action I take, the outcome will be the worst among all outcomes that might occur, given my action. Therefore, I will choose an action for which that worst outcome is best for me." In a two-player game, this reasoning leads player 1 to choose a solution to the problem

$$\max_{a^1 \in A^1}[\min_{a^2 \in A^2} u^1(a^1, a^2)]$$

and player 2 to choose a solution to the problem

$$\max_{a^2 \in A^2}[\min_{a^1 \in A^1} u^2(a^1, a^2)].$$

The maximum in each case is the highest payoff that each player can guarantee for herself.

Consider, for example, the following variant of Bach or Stravinsky.

	B	S
B	$2,1$	$0.5,0.5$
S	$0,0$	$1,2$

The game has two Nash equilibria, (B,B) and (S,S). Suppose that each player chooses an action using the pessimistic reasoning we have described. If player 1 chooses B, then the worst outcome for her is (B,S), and if she chooses S, the worse outcome is (S,B). The former is better than the latter for her, so she chooses B. Similarly, player 2 chooses S. Thus, if the two players reason in this way, the outcome is (B,S) (and the players do not meet).

Consider now the location game of Example 15.8. This game, as we noted, is strictly competitive: whenever the market share of one player increases, the market share of the other player decreases. The game has a unique Nash equilibrium, in which both players choose the middle point, $\frac{1}{2}$. A player who chooses this location guarantees that her market share is at least $\frac{1}{2}$. If she chooses any other location, then if the other player chooses a point between the middle point and her point, she gets less than half the market. For example, if a player chooses 0.6 then she gets less than half the market if the other player chooses 0.55. Thus for this game, unlike the variant of Bach or Stravinsky, which is not strictly competitive, Nash equilibrium and the pessimistic reasoning we have described lead to the same conclusion. We now show that the same is true for any strictly competitive game, and hence if a strictly competitive game has more than one equilibrium then each player's payoff in every equilibrium is the same.

Proposition 15.3: Maxminimization and Nash equilibrium

Let $G = \langle \{1,2\}, (A^i)_{i\in\{1,2\}}, (\succsim^i)_{i\in\{1,2\}} \rangle$ be a strictly competitive game. (i) If (a^1, a^2) is a Nash equilibrium of G, then for each player i, a^i is a solution of $\max_{x^i \in A^i} [\min_{x^j \in A^j} u^i(x^1, x^2)]$, where u^i represents $\succsim^i$ and j is the other player. (ii) If G has a Nash equilibrium, then each player's payoff is the same in all equilibria.

Proof

(i) Assume that (a^1, a^2) is a Nash equilibrium of G. If player 2 chooses an action different from a^2, her payoff is not higher than $u^2(a^1, a^2)$, so that player 1's payoff is not lower than $u^1(a^1, a^2)$. Thus the lowest payoff player 1 obtains if she chooses a^1 is $u^1(a^1, a^2)$:

$$u^1(a^1, a^2) = \min_{x^2 \in A^2} u^1(a^1, x^2).$$

Hence, by the definition of a maximizer,

$$u^1(a^1, a^2) \leq \max_{x^1 \in A^1} \min_{x^2 \in A^2} u^1(x^1, x^2).$$

Now, given that (a^1, a^2) is a Nash equilibrium of G, $u^1(a^1, a^2) \geq u^1(x^1, a^2)$ for all $x^1 \in A^1$. Thus $u^1(a^1, a^2) \geq \min_{x^2 \in A^2} u^1(x^1, x^2)$ for all $x^1 \in A^1$, and hence

$$u^1(a^1, a^2) \geq \max_{x^1 \in A^1} \min_{x^2 \in A^2} u^1(x^1, x^2).$$

We conclude that

$$u^1(a^1, a^2) = \max_{x^1 \in A^1} \min_{x^2 \in A^2} u^1(x^1, x^2).$$

(ii) That player 1's payoff is the same in all equilibria follows from (i).

Note that the result does not claim that a Nash equilibrium exists. Indeed, we have seen that the game odds or evens, which is strictly competitive, has no Nash equilibrium.

By Proposition 15.3, a player's payoff in a Nash equilibrium of a strictly competitive game is the maximum payoff the player can guarantee. We show now that it is also the lowest payoff the other player can inflict on her. As we noted earlier, we can take player 2's payoff function to be the negative of player 1's, ($u^2(x^1, x^2) = -u^1(x^1, x^2)$ for all (x^1, x^2)) and thus by Proposition 15.3, if (a^1, a^2) is a Nash equilibrium of the game then

$$-u^1(a^1,a^2) = u^2(a^1,a^2) = \max_{x^2\in A^2}\min_{x^1\in A^1}(-u^1(x^1,x^2))$$
$$= \max_{x^2\in A^2}(-\max_{x^1\in A^1}(u^1(x^1,x^2))) = -\min_{x^2\in A^2}\max_{x^1\in A^1} u^1(x^1,x^2),$$

so that

$$u^1(a^1,a^2) = \max_{x^1\in A^1}\min_{x^2\in A^2} u^1(x^1,x^2) = \min_{x^2\in A^2}\max_{x^1\in A^1} u^1(x^1,x^2).$$

That is, if the game has a Nash equilibrium then the maximum payoff a player can guarantee is the same as the lowest payoff the other player can inflict on her. For a game that is not strictly competitive, this equality does not generally hold, but the maximum payoff that a player can guarantee is never higher than the minimum that the other player can inflict on her (see Problem 12).

15.6 Kantian equilibrium

Nash equilibrium is the most commonly used solution concept for strategic games, but it is not the only possible solution concept. We now briefly discuss one alternative concept, Kantian equilibrium.

At a Nash equilibrium, no player wants to deviate under the assumption that the other players will not change their actions. At a Kantian equilibrium, no player wants to deviate under the assumption that if she does so, the other players will change their actions in the same way as she has. To complete the definition we need to specify the meaning of "the same way".

We illustrate the concept with a simple example. Consider a two-player game in which each player i's set of actions is $(0,1]$ and her preferences are represented by u^i. Assume that a player who considers deviating from an action pair, increasing or decreasing her action by a certain percentage, imagines that the other player will change her action in the same direction, by the same percentage. In equilibrium no player wishes to change her action under this assumption about the resulting change in the other player's action. Formally, (a^1,a^2) is a *Kantian equilibrium* if $u^i(a^1,a^2) \geq u^i(\lambda a^1,\lambda a^2)$ for $i = 1, 2$ and for all $\lambda > 0$.

We calculate the Kantian equilibrium for the quantity-setting duopoly in Example 15.10 with $c = 0$. For (a^1,a^2) to be a Kantian equilibrium of this game we need

$$u^1(a^1,a^2) = a^1(1-a^1-a^2) \geq \max_{\lambda>0} \lambda a^1(1-\lambda a^1-\lambda a^2)$$

and similarly for player 2. The solution of the maximization problem is $\lambda^* = 1/(2(a^1+a^2))$. For equilibrium we need $\lambda^* = 1$, so that $a^1+a^2 = \frac{1}{2}$. The condition for player 2 is identical, so any pair (a^1, a^2) for which $a^1 + a^2 = \frac{1}{2}$ is a Kantian equilibrium.

By contrast, the game has a unique Nash equilibrium, $(\frac{1}{3}, \frac{1}{3})$. So the total output produced in a Kantian equilibrium is less than the total output produced in a Nash equilibrium. The reason that $(\frac{1}{4}, \frac{1}{4})$ is not a Nash equilibrium is that an increase in output by a single player, assuming the other player does not change her output, is profitable. It is a Kantian equilibrium because an increase in output by a single player from $\frac{1}{4}$ is not profitable if it is accompanied by the same increase in the other player's output.

15.7 Mixed strategies

Consider the game matching pennies, specified as follows.

	H	T
H	$0,1$	$1,0$
T	$1,0$	$0,1$

As we have seen, this game has no Nash equilibrium.

Imagine that two large populations of individuals play the game, members of population 1 playing the role of player 1 and members of population 2 playing the role of player 2. From time to time, an individual is drawn randomly from each population and these two individuals play the game. Each individual in each population chooses the same action whenever she plays the game, but the individuals within each population may choose different actions. When two individuals are matched to play the game, neither of them knows the identity of the other player. We are interested in steady states in which each individual's belief about the distribution of actions in the other population is correct (perhaps because of her long experience playing the game) and each individual chooses her best action given these beliefs.

An implication of the game's not having a Nash equilibrium is that no configuration of choices in which all members of each population choose the same action is stable. For example, the configuration in which every individual in population 1 chooses T and every individual in population 2 chooses H is not consistent with a stable steady state because every individual in population 2, believing that she certainly faces an opponent who will choose T, is better off choosing T.

Now consider the possibility that some individuals in each population choose H and some choose T. Denote by p^H the fraction of individuals in population 1 who choose H. Then an individual in population 2 gets a payoff of 1 with probability p^H if she chooses H and with probability $1 - p^H$ if she chooses T. Thus if $p^H > \frac{1}{2}$ then every individual in population 2 prefers H to T, in which case the individuals in population 1 who choose H are not acting optimally. Hence the game has no steady state with $p^H > \frac{1}{2}$. Similarly, it has no steady state with $p^H < \frac{1}{2}$.

What if $p^H = \frac{1}{2}$? Then every individual in population 2 is indifferent between H and T: both actions yield the payoff 1 with probability $\frac{1}{2}$. Thus any distribution of actions among the individuals in population 2 is consistent with each of these individuals acting optimally. In particular, a distribution in which half the individuals choose each action is consistent. And given that distribution, by the same argument every individual in population 1 is indifferent between H and T, so that in particular half of them choosing H and half choosing T is consistent with each individual in population 1 choosing her action optimally, given the distribution of actions in population 2. In summary, every individual's action is optimal given the distribution of actions in the other population if and only if each action is chosen by half of each population.

In the remainder of the chapter we identify population i with player i and refer to a distribution of actions in a population as a mixed strategy. (The terminology "mixed strategy" relates to another interpretation of equilibrium, in which a player chooses a probability distribution over her actions. We do not discuss this interpretation.)

Definition 15.6: Mixed strategy

Given a strategic game $\langle N, (A^i)_{i \in N}, (\succsim^i)_{i \in N} \rangle$, a *mixed strategy* for player i is a probability distribution over A^i. A mixed strategy α^i that is concentrated on one action (i.e. $\alpha^i(a^i) = 1$ for some $a^i \in A^i$) is a *pure strategy*.

If A^i consists of a finite (or countable) number of actions, a mixed strategy α^i of player i assigns a nonnegative number $\alpha^i(a^i)$ to each $a^i \in A^i$, and the sum of these numbers is 1. We interpret $\alpha^i(a^i)$ as the proportion of population i that chooses the action a^i.

If some players' mixed strategies are not pure, players face uncertainty. To analyze their choices, we therefore need to know their preferences over lotteries over action profiles, not only over the action profiles themselves. Following convention we adopt the expected utility approach (see Section 3.3) and assume that the preferences of each player i over lotteries over action profiles are represented by the expected value of some function u^i that assigns a number to each action profile. *Thus in the remainder of this section, in Section 15.9, and in the exercises for these sections, we specify the preferences of each player i in a strategic game by giving a Bernoulli function u^i whose expected value represents the player's preferences over lotteries over action profiles* (rather than a preference relation over action profiles).

We now define a concept of equilibrium in the spirit of Nash equilibrium in which the behavior of each player is described by a mixed strategy rather than an action. We give a definition only for games in which the number of actions of each player is finite or countably infinite. A definition for games with more general action sets is mathematically more subtle.

Definition 15.7: Mixed strategy equilibrium of strategic game

Let $G = \langle N, (A^i)_{i\in N}, (u^i)_{i\in N}\rangle$ be a strategic game for which the set A^i of actions of each player i is finite or countably infinite. A profile $(\alpha^i)_{i\in N}$ of mixed strategies is a *mixed strategy equilibrium* of G if for every player $i \in N$ and every action $a^i \in A^i$ for which $\alpha^i(a^i) > 0$, i's expected payoff (according to u^i) from a^i given α^{-i} is at least as high as her expected payoff from x^i given α^{-i} for any $x^i \in A^i$.

The notion of mixed strategy equilibrium extends the notion of Nash equilibrium in the sense that (*i*) any Nash equilibrium is a mixed strategy equilibrium in which each player's mixed strategy is a pure strategy and (*ii*) if α is a mixed strategy equilibrium in which each player's mixed strategy is pure, with $\alpha^i(a^i) = 1$ for every player i, then $(a^i)_{i\in N}$ is a Nash equilibrium.

Although not every strategic game has a Nash equilibrium, every game in which each player's set of actions is finite has a mixed strategy equilibrium. A proof of this result is beyond the scope of this book.

Example 15.13: Mixed strategy equilibrium in matching pennies

	H	T
H	$0,1$	$1,0$
T	$1,0$	$0,1$

Let (α^1, α^2) be a pair of mixed strategies. Let $p^1 = \alpha^1(H)$ and $p^2 = \alpha^2(H)$. The pair is not a mixed strategy equilibrium if $\alpha^i(a^i) = 1$ for some action a^i, for either player i. If $p^1 = 1$, for example, then the only optimal action of player 2 is H but if $p^2 = 1$ then player 1's action H is not optimal.

For (α^1, α^2) to be an equilibrium with $p^1 \in (0,1)$, both actions of player 1 must yield the same expected payoff, so that $1 - p^2 = p^2$, and hence $p^2 = \frac{1}{2}$. The same consideration for player 2 implies that $p^1 = \frac{1}{2}$. Thus the only mixed strategy equilibrium of the game is the mixed strategy pair in which half of each population chooses each action.

The next example shows that a game that has a Nash equilibrium may have also mixed strategy equilibria that are not pure.

Example 15.14: Mixed strategy equilibria of Bach or Stravinsky

	B	S
B	2,1	0,0
S	0,0	1,2

The game has two Nash equilibria, (B,B) and (S,S). Now consider mixed strategy equilibria (α^1,α^2) in which at least one of the strategies is not pure. If one player's mixed strategy is pure, the other player's optimal strategy is pure, so in any non-pure equilibrium both players assign positive probability to both of their actions.

For both actions to be optimal for player 1 we need the expected payoffs to these actions, given player 2's mixed strategy, to be equal. Thus we need $2\alpha^2(B) = \alpha^2(S)$, so that $\alpha^2(B) = \frac{1}{3}$ and $\alpha^2(S) = \frac{2}{3}$. Similarly $\alpha^1(B) = \frac{2}{3}$ and $\alpha^1(S) = \frac{1}{3}$.

Hence the game has three mixed strategy equilibria, two that are pure and one that is not. In the non-pure equilibrium the probability of the players' meeting is $\frac{4}{9}$ and each player's expected payoff is $\frac{2}{3}$, less than the payoff from her worst pure equilibrium.

An interpretation of the non-pure equilibrium is that in each population two-thirds of individuals choose the action corresponding to their favorite outcome and one-third compromise.

We end the discussion of mixed strategies with a somewhat more complicated economic example.

Example 15.15: War of attrition

Two players compete for an indivisible object whose value is 1 for each player. Time is discrete, starting at period 0. In each period, each player can either give up or fight. The game ends when a player gives up, in which case the object is obtained by the other player. If both players give up in the same period, no one gets the object. For each period that passes before a player gives up, she incurs the cost $c \in (0,1)$. If player 1 plans to give up in period t and player 2 plans to do so in period s, with $s > t$, then player 2 gets the object in period $t+1$, incurring the cost $(t+1)c$, and player 1 incurs the cost tc.

We model the situation as a strategic game in which a player's action specifies the period in which she plans to give up. We have $N = \{1,2\}$ and, for $i = 1, 2$, $A^i = \{0,1,2,3,\ldots\}$ and

$$u^i(t^i, t^j) = \begin{cases} -t^i c & \text{if } t^i \leq t^j \\ 1-(t^j+1)c & \text{if } t^i > t^j. \end{cases}$$

First consider pure Nash equilibria. For any action pair (t^1, t^2) in which $t^1 > 0$ and $t^2 > 0$, the player who gives up first (or either player, if $t^1 = t^2$) can increase her payoff by deviating to give up immediately. Thus any pure Nash equilibrium has the form $(0, t^2)$ or $(t^1, 0)$. In any equilibrium, t^i is large enough that $c(t^i + 1) \geq 1$, so that player j's payoff from giving up immediately, 0, is at least $1 - (t^i + 1)c$, her payoff from waiting until period $t^i + 1$.

Let (α, α) be a symmetric mixed strategy equilibrium. (We leave the asymmetric mixed strategy equilibria for you to investigate.)

Step 1 *The set of periods to which α assigns positive probability has no holes: there are no numbers t_1 and t_2 with $t_1 < t_2$ for which $\alpha(t_1) = 0$ and $\alpha(t_2) > 0$.*

Proof. Suppose $\alpha(t_1) = 0$ and $\alpha(t_2) > 0$. Then there is a period t such that $\alpha(t) = 0$ and $\alpha(t+1) > 0$. We show that the action $t+1$ is not a best response to α. If the other player plans to give up before t, then t and $t+1$ yield the same payoff. If the other player plans to give up at $t+1$ or later, which happens with positive probability given that $\alpha(t+1) > 0$, then the player saves c by deviating from $t+1$ to t. ◁

Step 2 *We have $\alpha(t) > 0$ for all t.*

Proof. Suppose, to the contrary, that T is the last period with $\alpha(T) > 0$. Then T is not a best response to α since $T+1$ yields a higher expected payoff. The two actions yield the same outcome except when the other player plans to give up at period T, an event with positive probability $\alpha(T)$. Therefore the expected payoff of $T+1$ exceeds that of T by $\alpha(T)(1-c)$, which is positive given our assumption that $c < 1$. ◁

Step 3 *For every value of t we have $\alpha(t) = c(1-c)^t$.*

Proof. From the previous steps, a player's expected payoff to T and $T+1$ is the same for every period T. The following table gives player i's payoffs to these actions for each possible action t_j of player j.

	$t_i = T$	$t_i = T+1$
$t_j < T$	$1-(t_j+1)c$	$1-(t_j+1)c$
$t_j = T$	$-Tc$	$1-(T+1)c$
$t_j > T$	$-Tc$	$-(T+1)c$

The expected payoffs of player i from T and $T+1$ must be equal, given the mixed strategy α of player j, so that the difference between her expected payoffs must be 0:

$$\alpha(T)\big(1-(T+1)c-(-Tc)\big)+\left(1-\sum_{t=0}^{T}\alpha(t)\right)\big(-(T+1)c-(-Tc)\big)=0$$

or

$$\alpha(T)-c\left(1-\sum_{t=0}^{T-1}\alpha(t)\right)=0.$$

Thus for all T, conditional on not conceding before period T, the strategy α concedes at T with probability c, so that $\alpha(T)=c(1-c)^T$. ◁

15.8 Interpreting Nash equilibrium

The concept of Nash equilibrium has several interpretations. In this book we interpret an equilibrium to be a stable norm of behavior, or a convention. A Nash equilibrium is a profile of modes of behavior that is known to all players and is stable against the possibility that one of them will realize that her action is not optimal for her given the other players' behavior. Thus, for example, the equilibrium (Y,Y) in the coordination game represents the convention that the players meet at Y; the equilibrium (F,N) in Bach or Stravinsky represents the norm that player 1 (perhaps the younger player) always insists on meeting at her favorite concert whereas player 2 (the older player) yields; and the game matching pennies has no stable norm.

In a related interpretation, we imagine a collection of populations, one for each player. Whenever the game is played, one individual is drawn randomly from each population i to play the role of player i in the game. Each individual bases her decision on her beliefs about the other players' actions. In equilibrium these beliefs are correct and the action of each player in each population is optimal given the common expectation of the individuals in the population about the behavior of the individuals in the other populations.

As discussed in Section 15.7, this interpretation is appealing in the context of mixed strategy equilibrium. In that case, the individuals in each population may differ in their behavior. All individuals are anonymous, so that no individual obtains information about the action chosen by any specific individual. But every individual holds correct beliefs about the distribution of behavior in each of the other populations. A Nash equilibrium is a steady state in which every individual's belief about the action chosen by the individuals in each population is correct and any action assigned positive probability is optimal given the equilibrium distribution of actions in the other populations. Thus, for example, the mixed strategy equilibrium in matching pennies represents a steady state in which half of each population of individuals chooses each action.

Nash equilibrium is sometimes viewed as the outcome of a reasoning process by each player or as the outcome of an evolutionary process. In this book, we do not discuss these ideas; we focus on Nash equilibrium as a norm of behavior or as a steady state in the interaction between populations of individuals that frequently interact.

15.9 Correlated equilibrium

We have discussed some interpretations of mixed strategy equilibrium. We now briefly discuss an equilibrium concept that springs from another interpretation of mixed strategy equilibrium: each player bases her action on the realization of some private information that is known only to her, does not affect her preferences, and is independent of the information on which the other players base their actions.

Consider the game Bach or Stravinsky, reproduced here.

	B	S
B	2, 1	0, 0
S	0, 0	1, 2

Suppose that each player independently wakes up in a good mood with probability $\frac{1}{3}$ and in a bad mood with probability $\frac{2}{3}$. Then the mixed strategy equilibrium can be thought of as the result of each player's choosing the action she likes least (S for player 1, B for player 2) if and only if she wakes up in a good mood.

We generalize this idea by assuming that the signals on which the players base their actions may be correlated. Suppose, for example, that the weather has three equally likely states, x (rainy), y (cloudy), and z (clear), and

player 1 is in a bad mood in $\{x, y\}$ and in a good mood in z

player 2 is in a bad mood in $\{y, z\}$ and in a good mood in x.

Assume that each player knows only her own mood, not the other player's mood. Suppose that each player chooses her less favored action when her mood is good and her favored action when her mood is bad. Over time she accumulates information about the other player's behavior conditional on her mood. Then player 1 concludes that if she is in a bad mood, player 2 chooses B and S with equal probabilities. Given these beliefs she optimally chooses B, yielding expected payoff 1, which is greater than her expected payoff of choosing S, namely $\frac{1}{2}$. When she is in a good mood, she concludes that player 2 chooses S, making her choice of S optimal. Analogously, player 2's plan is optimal whatever she observes. Thus this behavior is an equilibrium in the sense that for each player, each signal she can receive, and the statistics about the other player's behavior given her signal, a player does not want to revise her rule of behavior.

Generalizing this idea leads to the following definition.

Definition 15.8: Correlated equilibrium of strategic game

Let $G = \langle N, (A^i)_{i \in N}, (u^i)_{i \in N} \rangle$ be a strategic game for which the set A^i of actions of each player i is finite. A candidate for a correlated equilibrium is a tuple $(\Omega, \mu, (P^i)_{i \in N}, (s^i)_{i \in N})$ for which

- Ω is a finite set (of *states*)
- μ is a probability measure on Ω
- for each player $i \in N$, P^i is a partition of Ω (i's *information partition*: if the state is $\omega \in \Omega$ then i is informed of the cell of P^i that includes ω)
- for each player $i \in N$, s^i is a function that assigns an action in A^i to each state in Ω such that the same action is assigned to all states in the same cell of P^i.

The tuple $(\Omega, \mu, (P^i), (s^i))$ is a *correlated equilibrium* if for every $\omega \in \Omega$ and each player i, the action $s^i(\omega)$ is a best response for i to the distribution of a^{-i} given the cell in P^i that contains ω.

Consider again the game Bach or Stravinsky. The correlated equilibrium that we have discussed in which the set of states is $\{x, y, z\}$ yields the distribution of outcomes that assigns equal probabilities to the three outcomes (B, B), (B, S), and (S, B):

	B	S
B	$\frac{1}{3}$	$\frac{1}{3}$
S	0	$\frac{1}{3}$

This distribution can be obtained also by another correlated equilibrium, defined as follows.

- The set of states is the set of outcomes, $\{(B,B),(B,S),(S,B),(S,S)\}$.
- The probability measure on this set assigns probability $\frac{1}{3}$ to each of the three states (B,B), (B,S), and (S,S).
- Player 1's information partition is $\{\{(B,B),(B,S)\},\{(S,B),(S,S)\}\}$ and player 2's is $\{\{(B,B),(S,B)\},\{(B,S),(S,S)\}\}$.
- Player 1's strategy in state (X,Y) chooses X and player 2's strategy chooses Y.

In this equilibrium, a state can be interpreted as the profile of actions recommended by nature, with each player being informed only of the action she is recommended to take.

The construction of this correlated equilibrium illustrates a general result: for every correlated equilibrium there is another correlated equilibrium with the same distribution of outcomes in which the set of states is the set of outcomes in the game.

Definition 15.9: Standard correlated equilibrium

Let $G = \langle N,(A^i)_{i\in N},(u^i)_{i\in N}\rangle$ be a strategic game. A *standard correlated equilibrium* is a correlated equilibrium $(\Omega,\mu,(P^i)_{i\in N},(s^i)_{i\in N})$ in which

- the set Ω of states is the set of outcomes (action profiles), $A = \times_{i\in N}A^i$
- the information partition P^i of each player i is the collection of all sets $\{(x^j)_{j\in N} : x^i = a^i\}$ for $a^i \in A^i$
- the strategy s^i of each player i is defined by $s^i((x^j)_{j\in N}) = x^i$.

The next proposition implies that if we are interested only in the distribution of outcomes in correlated equilibria then we can limit attention to standard correlated equilibria.

Proposition 15.4: Correlated and standard correlated equilibrium

For any correlated equilibrium there is a standard correlated equilibrium that induces the same distribution of outcomes.

Proof

Let $(\Omega,\mu,(P^i),(s^i))$ be a correlated equilibrium. For each player i let Q^i be the partition of Ω for which for each action $a^i \in A^i$ for which $s^i(\omega) = a^i$ for some $\omega \in \Omega$, there is a cell in Q^i that is the union of the cells in P^i to which

s^i assigns a^i. Then $(\Omega,\mu,(Q^i),(s^i))$ is a correlated equilibrium. The reason is a basic property of expected utility: if a^i is optimal given a set of cells in P^i then it is optimal also given the union of the set of cells.

To define the associated standard correlated equilibrium we need only specify the probability measure μ^* over the set of states, which is the set A of outcomes of the game. We define $\mu^*((a^i)_{i\in N}) = \mu(\{\omega \in \Omega : s^i(\omega) = a^i \text{ for all } i \in N\}$. In this standard correlated equilibrium a player's signal is the action she is supposed to take. Given that she is supposed to choose a^i, her belief about the other players' actions is the same as it is in $(\Omega,\mu,(Q^i),(s^i))$ when she plays a^i, and is thus optimal.

Finally, every mixed strategy equilibrium can be described also as a correlated equilibrium. For example, the game Bach or Stravinsky has a mixed strategy equilibrium (α^1,α^2) for which $\alpha^1(B) = \alpha^2(S) = \frac{2}{3}$. The behavior presented by this equilibrium is obtained also by a standard correlated equilibrium with $\mu(X,Y) = \alpha^1(X)\alpha^2(Y)$ (see the table).

	B	S
B	$\frac{2}{9}$	$\frac{4}{9}$
S	$\frac{1}{9}$	$\frac{2}{9}$

Given μ, whatever recommendation player 1 receives, she believes that player 2 chooses B and S with probabilities $\alpha^2(B)$ and $\alpha^2(S)$, so that both actions are optimal for her given her beliefs. Similarly for player 2.

More generally, if $(\alpha^i)_{i\in N}$ is a mixed strategy equilibrium of G then in the standard correlated equilibrium defined by $\mu((a^i)_{i\in N}) = \prod_{i\in N} \alpha^i(a^i)$, the players' choices are independent and each player's distribution of choices is the same as her choice in the mixed strategy equilibrium.

15.10 $S(1)$ equilibrium

We end the chapter with a discussion of another solution concept for finite strategic games. By doing so we wish to emphasize that Nash equilibrium, with or without mixed strategies, is not the only possible solution concept for strategic games.

At the heart of the concept of an $S(1)$ equilibrium lies an assumption about the procedure a player uses to decide the action to take when she is not familiar with the consequences of the possible actions. Imagine a large society in which each individual has to decide between two actions, L and R. Suppose that we know that an individual's experience from the action L is with equal probabilities

Very good or *Bad* and her experience from the action R is with equal probabilities *Good* or *Very bad*. The outcome is uncertain, so different individuals may have different experiences from using the actions. A new individual who arrives into the society does not have any idea about the virtues of the two alternatives, so she consults one individual who chose L and one who chose R. She compares their experiences and chooses accordingly: if either (*i*) the individual who chose L had a *Very good* experience or (*ii*) this individual had a *Bad* experience and the individual who chose R had a *Very bad* experience, then she chooses L, and otherwise she chooses R. Thus as observers we will find that a newcomer to the society chooses L with probability $\frac{3}{4}$ and R with probability $\frac{1}{4}$.

Let us turn back to games. We consider only two-player symmetric games. Such a game is characterized by a set Y of actions and a payoff function $u : Y \times Y \to \mathbb{R}$, with the interpretation that $u(a,b)$ is a player's payoff if she chooses a and the other player chooses b. For any action a and any mixed strategy σ (interpreted as a distribution of actions in the population), define $L(a,\sigma)$ to be the lottery that yields $u(a,b)$ with probability $\sigma(b)$ for each action b. We imagine that a player who enters the society samples each action once herself, or, for each possible action, asks an individual who chose that action about her experience. This information leads her to associate a payoff with each action, and she chooses the action with the highest payoff. Thus she selects the action a whenever $L(a,\sigma)$ yields a higher payoff than do all lotteries $L(x,\sigma)$ for $x \in Y \setminus \{a\}$. If more than one lottery yields the highest payoff, she chooses each of the tied actions with equal probabilities. Denote the probability that she chooses a by $W(a,\sigma)$. We define an $S(1)$ *equilibrium* to be a mixed strategy σ for which the probability $W(a,\sigma)$ is equal to $\sigma(a)$ for all $a \in Y$.

Definition 15.10: $S(1)$ equilibrium

Let $G = \langle\{1,2\},(A^i)_{i\in N},(u^i)_{i\in N}\rangle$ with $A^1 = A^2 = Y$ and, for all $a \in Y$ and $b \in Y$, $u^1(a,b) = u(a,b)$ and $u^2(a,b) = u(b,a)$, be a two-player symmetric strategic game. An $S(1)$ *equilibrium* of G is a mixed strategy σ for which the probability $W(a,\sigma)$ is equal to $\sigma(a)$ for all $a \in Y$.

Thus an $S(1)$ equilibrium is a stable distribution of play in the population: the distribution of the actions chosen by new entrants is equal to the equilibrium distribution.

Obviously, every strict symmetric Nash equilibrium, where all players choose some action a^*, is an $S(1)$ equilibrium: when she samples a^*, a new individual has a better experience than she does when she samples any other action, given that the other player chooses a^*.

Every finite symmetric strategic game has an $S(1)$ equilibrium. The proof of this result is above the level of this book, but we present it for readers who are familiar with Brouwer's fixed point theorem.

Proposition 15.5: Existence of $S(1)$ equilibrium

Every symmetric finite strategic game has an $S(1)$ equilibrium.

Proof

Assume that $Y = \{a_1, \ldots, a_K\}$ and let Δ be the set of all probability distributions over Y. The set Δ can be identified with the set of all K-vectors of nonnegative numbers that sum to 1, and is convex and compact. Define the function $F : \Delta \to \Delta$ by $F(\sigma) = (W(a_k, \sigma))_{k=1\ldots K}$. This function is continuous and so by Brouwer's fixed point theorem has at least one fixed point, namely a point σ^* for which $F(\sigma^*) = \sigma^*$. Any fixed point of F is an $S(1)$ equilibrium.

The next example demonstrates that the notion of $S(1)$ equilibrium, unlike that of mixed strategy equilibrium, depends only on the players' ordinal preferences over the set of action profiles.

Example 15.16: $S(1)$ equilibrium in simple game

Consider the following symmetric strategic game for $M > 3$.

	a	b
a	2	M
b	3	0

The game has no pure symmetric Nash equilibrium and has one symmetric mixed strategy equilibrium, (α, α) with $\alpha(a) = M/(M+1)$. This equilibrium depends on the value of M.

To calculate the $S(1)$ equilibrium note that a player concludes that a is the better action if (i) the other player chooses a when she samples a (payoff 2) and b when she samples b (payoff 0) or (ii) the other player chooses b when she samples a (payoff M). Thus for σ to be an $S(1)$ equilibrium we need $p = p(1-p)+(1-p)$, where $p = \sigma(a)$. This equation has a unique solution $p^* = (\sqrt{5}-1)/2 \approx 0.62$. Thus independently of M, as long as $M > 3$, the game has a unique $S(1)$ equilibrium, in which a is chosen with probability p^*.

The next example demonstrates that unlike a mixed strategy Nash equilibrium, an $S(1)$ equilibrium may assign positive probability to an action that is strictly dominated in the sense that another action yields a higher payoff regardless of the other player's action.

Example 15.17: Dominated action in support of $S(1)$ equilibrium

Consider the following game.

	a	b	c
a	2	5	8
b	1	4	7
c	0	3	6

A story behind this game is that each of two players holds 2 indivisible units that are worth 1 to her and 3 to the other player. Each player has to decide how many units she gives voluntarily to the other player: none (a), one (b), or two (c). Thus, for example, if a player keeps her two units and gets one unit from the other player her payoff is $2 \cdot 1 + 1 \cdot 3 = 5$.

The action a strictly dominates the other two, and the game has a unique Nash equilibrium, in which each player chooses a. To calculate the $S(1)$ equilibria, let $(\alpha, \beta, \gamma) = (\sigma(a), \sigma(b), \sigma(c))$. Then an $S(1)$ equilibrium is characterized by the following set of equations:

$$\begin{aligned} \alpha &= \alpha^3 + \beta(1-\gamma)^2 + \gamma \\ \beta &= \beta\alpha(1-\gamma) + \gamma(1-\gamma) \\ \alpha + \beta + \gamma &= 1. \end{aligned}$$

This set of equations has two solutions, $(\alpha, \beta, \gamma) = (1, 0, 0)$ and $(\alpha, \beta, \gamma) \approx (0.52, 0.28, 0.20)$. The first solution corresponds to the (strict) Nash equilibrium. The other solution assigns positive probabilities to b and c, even though these actions are strictly dominated.

If an action in a game is duplicated, the analysis of the Nash equilibria of the game is unaffected. The same is not true for the $S(1)$ equilibria, as the following example shows.

Example 15.18: Duplication of actions affects $S(1)$ equilibria

Consider the following games.

	a	b
a	1	4
b	3	2

	a_1	a_2	b
a_1	1	1	4
a_2	1	1	4
b	3	3	2

In the game on the right, the actions a_1 and a_2 are duplicates of a. The only $S(1)$ equilibrium of the game on the left assigns probability $\frac{1}{2}$ to each action. Denote by β the probability assigned to b. In an $S(1)$ equilibrium of the game on the right, $\beta = (1-\beta)^2$, which has a single solution, $\beta \approx 0.38$.

More generally, if the action a is replicated m times then the only $S(1)$ equilibrium assigns to b the probability β that is the solution of the equation $\beta = (1-\beta)^m$. As m increases without bound, this probability goes to 0. Thus, duplicating strategies has a significant affect on the $S(1)$ equilibrium.

Problems

Examples of games

1. *Centipede game*. Two players, 1 and 2, alternate turns in being able to stop interacting or to continue doing so. Player 1 starts the game. Initially each player has 0 in her account. Any decision by a player to continue reduces the player's account by 1 and adds 2 to the other player's account. After 100 actions to continue, the game stops. Thus, each player has at most 50 opportunities to stop the game. Each player wants the amount in her account at the end of the game to be as large as possible.

   Model this situation as a strategic game and show that the game has a unique Nash equilibrium.

2. *Demand game*. Two players can allocate ten indivisible desirable identical objects among themselves. Find the Nash equilibria of the following two games.

   a. The players simultaneously submit demands, members of $\{0, 1, \ldots, 10\}$. If the sum of the demands is at most 10, each player gets what she demands. Otherwise both get 0.

   b. As in part *a*, except that if the sum of the demands exceeds 10, then (*i*) if the demands differ then the player who demands less gets her demand and the other player gets the rest, and (*ii*) if the demands are the same then each player gets 5.

3. *War of attrition*. Two individuals, 1 and 2, compete for an object. Individual i's valuation of the object is v^i for $i = 1, 2$. Time is a continuous variable that starts at 0 and continues forever. The object is assigned to one of the individuals once the other one gives up. If both of them give up at the same

time, the object is divided equally (half the object is worth $\frac{1}{2}v^i$ to i). As long as neither individual gives up, each individual loses 1 unit of payoff per unit of time.

Model the situation as a strategic game and show that in every Nash equilibrium the game ends immediately.

4. *Extended Prisoner's dilemma.* Each of n tenants in a large building has to decide whether to keep her property clean, C, or not, D. Assume that each player's preferences can be represented by a payoff function in which she loses $B > 0$ if she keeps her property clean and loses $L > 0$ for every other tenant who chooses D. Model the situation as a strategic game and find the Nash equilibria for any parameters $n \geq 2$, B, and L.

5. *Guessing two-thirds of the average.* Each of n players has to name a member of $\{1,\ldots,100\}$. A player gets a prize if the number she names is the integer closest to two-thirds of the average number named by all players (or one of the two closest integers, if two integers are equally close). Notice that it is possible that nobody gets a prize or that several players get prizes.

   Model the situation as a strategic game and find its Nash equilibria.

6. *Cheap talk.* Two players are about to play Bach or Stravinsky (BoS, Example 15.3). Before doing so, player 1 sends one of the following messages to player 2: "I will choose B", or "I will choose S". Construct a strategic game in which an action of player 1 is a combination of the message to send and an action in BoS (a total of four possible actions) and an action for player 2 is a specification of the action in BoS to take for each possible message of player 1 (a total of four possible actions). Assume that both players care only about the payoff in BoS (not about the content of the message). Find the Nash equilibria of this game.

Economic games

7. *War.* Two players, 1 and 2, fight over a single indivisible object worth $V > 0$ to each of them. Each player invests in becoming more powerful; denote by e^i, a nonnegative number, the investment of player i. Given investments (e^1, e^2), player i's probability $p^i(e^1, e^2)$ of winning the object is $e^i/(e^i + e^j)$ if $e^1 + e^2 > 0$, and $\frac{1}{2}$ if $e^1 = e^2 = 0$. The preferences of each player i are represented by the payoff function $p^i(e^1, e^2)V - e^i$.

   Model the situation as a strategic game, show that in all Nash equilibria the two players choose the same investment, and characterize this investment level.

8. *All-pay auction.* An all-pay auction is a sealed-bid auction in which every bidder (not only the winner) pays her bid. Assume that there are two players, and that if their bids are the same each gets half of her value of the object.

   Model the situation as a strategic game and show that it does not have a Nash equilibrium.

9. *Another version of the location game.* Consider a variant of the location game in which the two players are candidates for a post and the set of positions is the interval $[0,1]$. A population of voters has favorite positions distributed uniformly over this interval; each voter endorses the candidate whose position is closer to her favorite position. (The fraction of citizens with favorite positions equidistant from the candidates' positions is zero, so we can ignore these citizens.) A candidate cares only about whether she receives more, the same number, or fewer votes than the other candidate.

   Model the situation as a strategic game and show that it has a unique Nash equilibrium.

10. *Nash demand game.* Two players bargain over one divisible unit of a good. Each player submits a demand, a number in $[0,1]$. For the pair of demands (t^1, t^2), the probability that agreement is reached is $g(t^1+t^2) \in [0,1]$, where g is differentiable, nonincreasing, positive when the sum of the demands is less than 1, and 0 when the sum of the demand exceeds 1. If agreement is reached on (t^1, t^2) then player i $(= 1, 2)$ gets t^i and her utility is $\alpha^i t^i$, with $\alpha^i > 0$. Each player maximizes her expected utility.

    a. Model the situation as a strategic game and show that in any Nash equilibrium the two players make the same demand.

    b. For any $\varepsilon > 0$, let G_ε be a game with $g_\varepsilon(t^1+t^2) = 1$ if $t^1+t^2 \leq 1-\varepsilon$. What can you say about the limit of the Nash equilibria of G_ε as $\varepsilon \to 0$?

11. *Contribution game.* Two players contribute to a joint project. The payoff function of player i has the form $v^i(c^1+c^2) - c^i$, where c^i is i's contribution and v^i is an increasing, differentiable, and concave function satisfying $v^i(0) = 0$. Assume that for each player i there is a number $x^i > 0$ for which $(v^i)'(x^i) = 1$ (so that $(v^i)'(0) > 1$ and hence each player optimally contributes a positive amount if the other player contributes zero). Finally assume that player 2 is interested in the project more than player 1 in the sense that $(v^2)'(x) > (v^1)'(x)$ for all x.

    Model the situation as a strategic game and show that in any Nash equilibrium only player 2's contribution is positive.

Strictly competitive games

12. max min *versus* min max. Show that in any two-player strategic game the maximum payoff a player can guarantee for herself is at most the minimum payoff that the other player can inflict on her.

13. *Comparative statics.* Consider two games G_1 and G_2 that differ only in that one of the payoffs for player 1 is higher in G_1 than it is in G_2.

    a. Show that if the games are strictly competitive then for any Nash equilibria of G_1 and G_2, player 1's payoff in the equilibrium of G_1 is at least as high as her payoff in the equilibrium of G_2.

    b. Give an example to show that the same is not necessarily true for games that are not strictly competitive.

Kantian equilibrium

14. *Kantian equilibrium.* Find the Kantian equilibrium (Section 15.6) for the price-setting duopoly in Example 15.11 with no production costs.

Mixed strategy equilibrium

15. *Mixed strategy equilibrium.* Find the mixed strategy equilibria of the following game.

	L	M	R
T	2,2	0,3	2,2
B	3,0	1,1	2,2

16. *Hawk or dove.* A population of individuals is frequently matched in pairs to fight over an object worth 1. Each individual can choose either Hawk (H) or Dove (D). If one individual chooses H and the other chooses D then the first individual gets the object. If both choose D then the object is split equally between the individuals. If both choose H then neither of them gets the object and each player i suffers a loss of $c^i > 0$. The situation is modeled by the following strategic game.

	H	D
H	$-c^1, -c^2$	1,0
D	0,1	0.5,0.5

The game has two Nash equilibria, (H, D) and (D, H). Prove that it has only one other mixed strategy equilibrium. Show that the higher is a player's loss when both players choose H the higher is her payoff in this equilibrium.

17. *Attack and defend.* Army 1 has one missile, which it can use to attack one of three targets of army 2. The significance of the three targets is given by the numbers $v(1) > v(2) > v(3) > 0$. The missile hits a target only if it is not protected by an anti-missile battery. Army 2 has one such battery. Army 1 has to decide which target to attack and army 2 has to decide which target to defend. If target t is hit then army 1's payoff is $v(t)$ and army 2's payoff is $-v(t)$; if no target is hit, each army's payoff is zero.

    a. Model this situation as a strategic game.

    b. Show that in any mixed strategy equilibrium, army 1 attacks both target 1 and target 2 with positive probability.

    c. Show that if $v(3) \leq v(2)v(1)/(v(1)+v(2))$ then the game has an equilibrium in which target 3 is not attacked and not defended.

18. *A committee.* The three members of a committee disagree about the best option. Members 1 and 2 favor option A, whereas member 3 favors option B. Each member decides whether to attend a meeting; if she attends, she votes for her favorite option. The option chosen is the one that receives a majority of the votes. If the vote is a tie (including the case in which nobody attends the meeting), each option is chosen with probability $\frac{1}{2}$. Each player's payoff depends on whether she attends the meeting and whether the outcome is the one she favors, as given in the following table.

	favored	not
participate	$1-c$	$-c$
not	1	0

Assume that $c < \frac{1}{2}$. Find the mixed strategy equilibria of the strategic game that models this situation in which players 1 and 2 (who both favor A) use the same strategy. Show that in such a mixed strategy equilibrium, A may be chosen with probability less than 1 and study how the equilibrium expected payoffs depend on c.

19. *O'Neill's game.* Each of two players chooses one of four cards labeled 2, 3, 4, and J. Player 1 wins if

    - the players choose different numbered cards (2, 3, or 4) or
    - both players choose J,

and otherwise player 2 wins. Model the situation as a strategic game and find the mixed strategy equilibria of the game.

20. *All-pay auction.* An item worth 10 is offered in an all-pay auction. Two players participate in the auction. Each player submits a monetary bid that is an integer between 0 and 10 and pays that amount regardless of the other player's bid. If one player's bid is higher than the other's, she receives the item. If the players' bids are the same, neither player receives the item. The players are risk neutral.

    a. Show that the game has no Nash equilibrium in pure strategies.

    b. Prove that the expected payoff for each player in any symmetric mixed strategy equilibrium is 0.

    c. Characterize the symmetric mixed strategy equilibria.

    d. Find an asymmetric mixed strategy equilibrium.

Correlated equilibrium

21. *Aumann's game.* Consider the following game.

	A	B
A	$6,6$	$2,7$
B	$7,2$	$0,0$

    Show that the game has a correlated equilibrium with a payoff profile that is not a convex combination of the payoff profiles of the three Nash equilibria (with and without mixed strategies).

22. *Convexity of the set of payoff vectors.* Show that the set of correlated equilibrium payoff profiles is convex. That is, if there are correlated equilibria that yield the payoff profiles $(u^i)_{i\in N}$ and $(v^i)_{i\in N}$ then for every $\lambda \in [0,1]$ there is also a correlated equilibrium with payoff profile $(\lambda u^i + (1-\lambda)v^i)_{i\in N}$.

$S(1)$ equilibrium

23. *$S(1)$ equilibrium of a 2×2 game.* Consider a symmetric two-player game in which the set of actions is $\{a,b\}$. Assume that a strictly dominates b: $u(a,x) > u(b,x)$ for $x = a,\ b$. Show that the only $S(1)$ equilibrium of the game is its unique Nash equilibrium.

24. *$S(1)$ equilibrium in price-setting duopoly.* Each of two sellers holds an indivisible unit of a good. Each seller chooses one of the K possible prices $p_1,\ldots,p_K$ with $0 < p_1 < \cdots < p_K$. The seller whose price is lower obtains a payoff equal to her price and the other seller obtains a payoff of 0. If the

prices are the same, each seller's payoff is half of the common price. Assume that $p_{k-1} > \frac{1}{2}p_k$ for all $k > 1$. Show that the game has a unique $S(1)$ equilibrium.

25. *$S(2)$ equilibrium.* The concept of $S(2)$ equilibrium is a variant of $S(1)$ equilibrium in which each player samples each action twice, rather than once, and chooses the action for which the average payoff for her two samples is highest. Compare the $S(1)$ and $S(2)$ equilibria of the following symmetric game.

	a	b
a	2	5
b	3	0

Notes

The model of a strategic game was developed by Borel (1921) and von Neumann (1928). The notion of Nash equilibrium is due to Nash (1950). (Cournot 1838, Chapter 7 is a precursor.)

Proposition 15.2 is a simple example of a result of Topkis (1979). The theory of strictly competitive games was developed by von Neumann and Morgenstern (1944). The notion of Kantian equilibrium is due to Roemer (2010). The notion of a mixed strategy was developed by Borel (1921, 1924, 1927). The notion of correlated equilibrium is due to Aumann (1974). Section 15.10, on $S(1)$ equilibrium, follows Osborne and Rubinstein (1998).

The Traveler's Dilemma (Example 15.1) is due to Basu (1994). The Prisoner's dilemma (Example 15.2) seems to have been first studied, in 1950, by Melvin Dresher and Merrill Flood (see Flood 1958/59). The game-theoretic study of auctions (Examples 15.6 and 15.7) was initiated by Vickrey (1961). The location game (Example 15.8) is due to Hotelling (1929). The model of quantity-setting producers (Example 15.10) is due to Cournot (1838) and the model of price-setting producers (Example 15.11) is named for Bertrand (1883). The war of attrition (Example 15.15) is due to Maynard Smith (1974).

The centipede game (Problem 1) is due to Rosenthal (1981), the game in Problem 5 is taken from Moulin (1986, 72), and the game in Problem 19 is due to O'Neill (1987). The game in Problem 21 is taken from Aumann (1974); it is the game he uses to demonstrate the concept of correlated equilibrium.