Models in Microeconomic Theory

Expanded Second Edition

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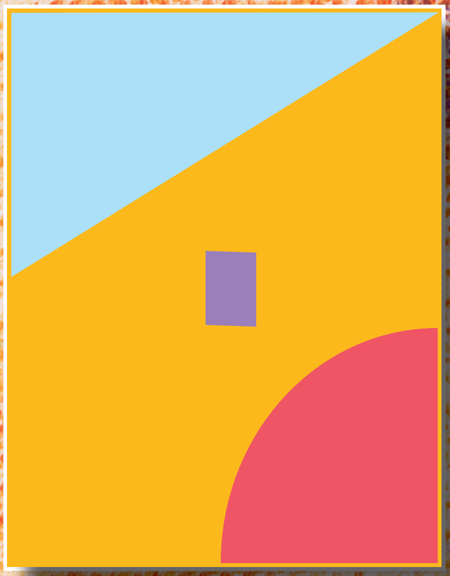

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4 Consumer preferences

In this chapter and the next we study preferences and choice in a context central to standard economic theory: an individual contemplating and choosing quantities of various goods. We refer to such an individual as a consumer. In this chapter, which is parallel to Chapter 1, we discuss preferences, without considering choice. In the next chapter, parallel to Chapter 2, we discuss properties of a consumer's choice function.

4.1 Bundles of goods

We take the set X of all alternatives that a consumer may face to be $\mathbb{R}^2_+$, the set of all pairs of nonnegative numbers. We refer to an element $(x_1, x_2) \in X$ as a bundle and interpret it as a pair of quantities of two goods, called 1 and 2.

Definition 4.1: Set of alternatives (bundles)

The set of alternatives is $X = \mathbb{R}^2_+$. A member of X is a *bundle*.

Goods could be entities like tables, potatoes, money, or leisure time. But, more abstractly, goods can be thought of as considerations the consumer has in mind; his preferences over X reflect his tradeoffs between these considerations. For example, the two goods could be the amounts of attention devoted to two projects or the welfare of the individual and his partner.

The assumption that $X = \mathbb{R}^2_+$ may seem odd, since talking about π tables or $\frac{1}{9}$ of a car has little meaning. We consider the quantities of the goods to be continuous variables for modeling convenience: doing so allows us to easily talk about the tradeoffs consumers face when they want more of each good but are constrained in what they can achieve.

The algebraic operations on the space $X = \mathbb{R}^2_+$ have natural interpretations. Given two bundles x and y, $x + y = (x_1 + x_2, y_1 + y_2)$ is the bundle formed by combining x and y into one bundle. Given a bundle x and a positive number λ, the bundle $\lambda x = (\lambda x_1, \lambda x_2)$ is the λ-multiple of the bundle x. For example, for any integer $m > 1$ the bundle $(1/m)x$ is the bundle obtained by dividing x into m equal parts. Note that given two bundles x and y and a number $\lambda \in (0, 1)$, the bundle $\lambda x + (1-\lambda)y$ lies on the line segment in $\mathbb{R}^2_+$ that connects the two bundles.

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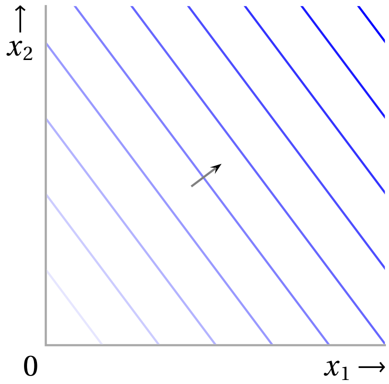
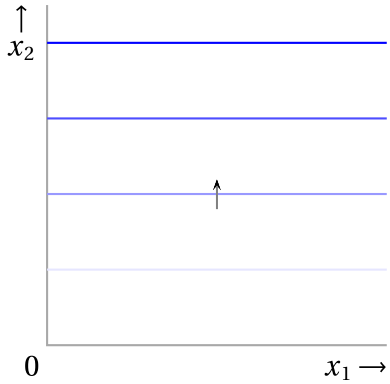


(a) Some indifference sets for the preference relation in Example 4.1 for $v_1/v_2 = \frac{4}{3}$.



(b) Some indifference sets for the preference relation in Example 4.2.

Figure 4.1

4.2 Preferences over bundles

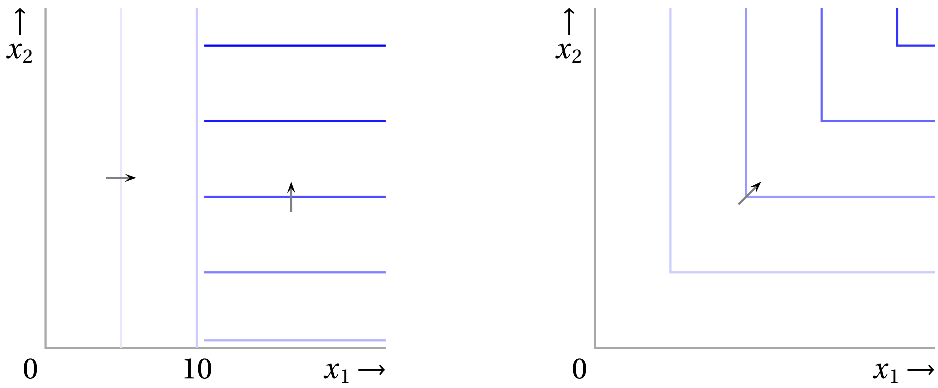
We now give some examples of preference relations over bundles. Many preference relations may helpfully be illustrated by diagrams that show a few indifference sets (sometimes called indifference curves). The indifference set for the preference relation $\succsim$ and bundle a is $\{y \in X : y \sim a\}$, the set of all bundles indifferent to a. The collection of all indifferent sets is the partition induced by the equivalence relation $\sim$. If $\succsim$ is represented by a utility function u, the indifference set for the bundle a can alternatively be expressed as $\{y \in X : u(y) = u(a)\}$, the contour of u for the bundle a.

Example 4.1: Constant tradeoff

The consumer has in mind two numbers v_1 and v_2, where v_i is the value he assigns to a unit of good i. His preference relation $\succsim$ is defined by the condition that $x \succsim y$ if $v_1x_1 + v_2x_2 \geq v_1y_1 + v_2y_2$. Thus $\succsim$ is represented by the utility function $v_1x_1 + v_2x_2$. The indifference set for the bundle (a_1, a_2) is $\{(x_1, x_2) : v_1x_1 + v_2x_2 = v_1a_1 + v_2a_2\}$, a line with slope $-v_1/v_2$. Figure 4.1a shows some indifference sets for $v_1/v_2 = \frac{4}{3}$. The arrow in the figure indicates the direction in which bundles are preferred.

Example 4.2: Only good 2 is valued

The consumer cares only about good 2, which he likes. His preference relation is represented by the utility function x_2. For this preference relation, every indifference set is a horizontal line; see Figure 4.1b.



(a) Some indifference sets for the preference relation in Example 4.3.

(b) Some indifference sets for the preference relation in Example 4.4.

Figure 4.2

Example 4.3: Minimal amount of good 1 and then good 2

The consumer cares only about increasing the quantity of good 1 until this quantity exceeds 10, and then he cares only about increasing the quantity of good 2. Precisely, $(x_1, x_2) \succsim (y_1, y_2)$ if (*i*) $y_1 \leq 10$ and $x_1 \geq y_1$ or (*ii*) $x_1 > 10$, $y_1 > 10$, and $x_2 \geq y_2$.

These preferences are represented by the utility function

$$\begin{cases} x_1 & \text{if } x_1 \leq 10 \\ 11 + x_2 & \text{if } x_1 > 10. \end{cases}$$

See Figure 4.2a. Notice that the indifference sets for utility levels above 10 are horizontal half lines that are open on the left.

Example 4.4: Complementary goods

The consumer wants the same amount of each good and prefers larger quantities. That is, he prefers a bundle x to a bundle y if and only if $\min\{x_1, x_2\} > \min\{y_1, y_2\}$. (Think of the goods as right and left shoes). Thus $\min\{x_1, x_2\}$ is a utility function that represents his preference relation (see Figure 4.2b).

Example 4.5: Ideal bundle

The consumer has in mind an ideal bundle x^*. He prefers a bundle x to a bundle y if and only if x is closer to x^* than is y according to some measure

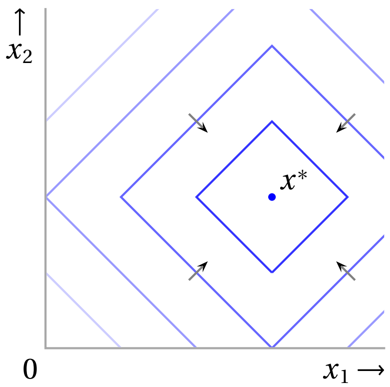


Figure 4.3 Some indifference sets for the preference relation in Example 4.5.

of distance. An example of a distance measure is the sum of the absolute differences of the components, in which case $x \succsim y$ if $|x_1 - x_1^*| + |x_2 - x_2^*| \leq |y_1 - x_1^*| + |y_2 - x_2^*|$. A utility function that represents this preference relation is $-(|x_1 - x_1^*| + |x_2 - x_2^*|)$. See Figure 4.3.

Example 4.6: Lexicographic preferences

The consumer cares primarily about the quantity of good 1; if this quantity is the same in two bundles, then he prefers the bundle with the larger quantity of good 2. Formally, $x \succsim y$ if either (*i*) $x_1 > y_1$ or (*ii*) $x_1 = y_1$ and $x_2 \geq y_2$. For this preference relation, for any two bundles x and y we have $x \succ y$ or $y \succ x$, so that each indifference set consists of a single point. The preference relation has no utility representation (Proposition 1.2).

In the rest of the chapter we discuss several properties of consumers' preferences that are often assumed in economic models.

4.3 Monotonicity

Monotonicity is a property of a consumer's preference relation that expresses the assumption that goods are desirable.

Definition 4.2: Monotone preference relation

The preference relation $\succsim$ on $\mathbb{R}^2_+$ is *monotone* if

$$x_1 \geq y_1 \text{ and } x_2 \geq y_2 \quad \Rightarrow \quad (x_1, x_2) \succsim (y_1, y_2)$$

and

$$x_1 > y_1 \text{ and } x_2 > y_2 \quad \Rightarrow \quad (x_1, x_2) \succ (y_1, y_2).$$

Thus if the bundle y is obtained from the bundle x by adding a positive amount of one of the goods then for a monotone preference relation $\succsim$ we have $y \succsim x$, and if y is obtained from x by adding positive amounts of both goods then $y \succ x$. For example, the bundle $(3,7)$ is preferred to the bundle $(2,6)$ and it may be preferred to $(3,5)$ or indifferent to it, but cannot be inferior.

The following property is a stronger version of monotonicity. If the bundle x has more of one good than the bundle y and not less of the other good then for a strongly monotone preference relation $\succsim$ we have $x \succ y$.

Definition 4.3: Strongly monotone preference relation

The preference relation $\succsim$ on $\mathbb{R}^2_+$ is *strongly monotone* if

$$x_1 \geq y_1,\ x_2 \geq y_2, \text{ and } (x_1,x_2) \neq (y_1,y_2) \quad \Rightarrow \quad (x_1,x_2) \succ (y_1,y_2).$$

The following table indicates, for each example in the previous section, whether the preference relation is monotone or strongly monotone.

Example	Monotonicity	Strong monotonicity
4.1: Constant tradeoff	✓	if $v_1 > 0$ and $v_2 > 0$
4.2: Only good 2 is valued	✓	✗
4.3: Minimal amount of 1, then 2	✓	✗
4.4: Complementary goods	✓	✗
4.5: Ideal bundle	✗	✗
4.6: Lexicographic	✓	✓

4.4 Continuity

Continuity is a property of a consumer's preference relation that captures the idea that if a bundle x is preferred to a bundle y then bundles close to x are preferred to bundles close to y.

Definition 4.4: Continuous preference relation

The preference relation $\succsim$ on $\mathbb{R}^2_+$ is *continuous* if whenever $x \succ y$ there exists a number $\varepsilon > 0$ such that for every bundle a for which the distance to x is less than ε and every bundle b for which the distance to y is less than ε we have $a \succ b$ (where the distance between any bundles (w_1,w_2) and (z_1,z_2) is $\sqrt{|w_1 - z_1|^2 + |w_2 - z_2|^2}$).

Note that a lexicographic preference relation is not continuous. We have $x = (1,2) \succ y = (1,0)$, but for every $\varepsilon > 0$ the distance of the bundle $a_\varepsilon = (1 - \varepsilon/2, 2)$ from x is less than ε but nevertheless $a_\varepsilon \prec y$.

Proposition 4.1: Continuous preference relation and continuous utility

A preference relation on $\mathbb{R}^2_+$ that can be represented by a continuous utility function is continuous.

Proof

Let $\succsim$ be a preference relation and let u be a continuous function that represents it. Let $x \succ y$. Then $u(x) > u(y)$. Let $\varepsilon = \frac{1}{3}(u(x) - u(y))$. By the continuity of u there exists $\delta > 0$ small enough such that for every bundle a within the distance δ of x and every bundle b within the distance δ of y we have $u(a) > u(x) - \varepsilon$ and $u(y) + \varepsilon > u(b)$. Thus for all such bundles a and b we have $u(a) > u(x) - \varepsilon > u(y) + \varepsilon > u(b)$ and thus $a \succ b$.

Comments

1. The converse result holds also: every continuous preference relation can be represented by a continuous utility function. A proof of this result is above the mathematical level of this book.
2. One can show that if $\succsim$ is a continuous preference relation on X and $a \succ b \succ c$ then on the line between the bundles a and c there is a bundle that is indifferent to b. That is, there is a number $0 < \lambda < 1$ such that $\lambda a + (1-\lambda)c \sim b$. This property is analogous to the property of continuity of preferences over the space of lotteries in the previous chapter.

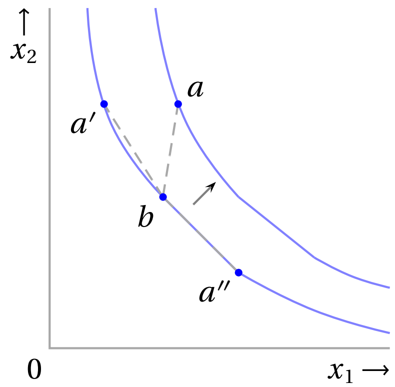
4.5 Convexity

Consider a world in which five candidates for a political job have positions commonly recognized to be ordered along the left-right political line as follows:

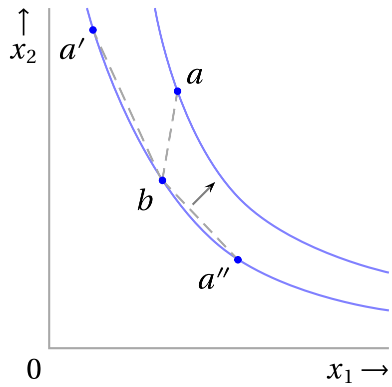
$$\text{——}\ D\ \text{—}\ A\ \text{———}\ C\ \text{—————}\ B\ \text{———}\ E\ \text{——}$$

Assume that a person tells you that he cares only about the candidates' positions on this dimension and says that he prefers A to B. What additional conclusions are you likely to make about his preferences?

You would probably conclude that he prefers C to B. If moving from B to A is an improvement, then going part of the way should also be an improvement. As to the comparison between A and C you would probably be unsure: you might think that he prefers A (if you believe that he is inclined to the left) or you might think that he prefers C (if you think that C is his favorite position among those adopted by the candidates). Thus our intuition is asymmetric: if a change makes



(a) Some indifference sets for a convex preference relation. Each bundle a, a', and a'' is at least as good as b, and all the bundles on each line segment from b to any one of these bundles are also at least as good as b, though bundles between a'' and b are not strictly better.



(b) Some indifference sets for a strictly convex preference relation. Each bundle a, a', and a'' is at least as good as b, and all the bundles on each line segment from b to any one of these bundles, excluding the endpoints, are better than b.

Figure 4.4

the person better off then a partial change probably does so too, but if a change makes him worse off then a partial change may make him better off.

Another natural conclusion is that a person who prefers A to B prefers also B to E, because it does not make sense that he considers candidates both to the left and to the right of B to be improvements over B. But D might be preferred to A (if D is the person's favorite candidate) or inferior to A (if A is the person's favorite candidate).

This example leads us to define a property of preferences called convexity, which is often assumed in economic theory.

Definition 4.5: Convex preference relation

The preference relation $\succsim$ on $\mathbb{R}^2_+$ is *convex* if

$$a \succsim b \quad \Rightarrow \quad \lambda a + (1-\lambda)b \succsim b \text{ for all } \lambda \in (0,1)$$

and is *strictly convex* if

$$a \succsim b \text{ and } a \neq b \quad \Rightarrow \quad \lambda a + (1-\lambda)b \succ b \text{ for all } \lambda \in (0,1).$$

Geometrically, $\lambda a + (1-\lambda)b$ is a bundle on the line segment from a to b, so the condition for a convex preference relation says that if a is at least as good as b then every bundle on the line segment from a to b is at least as good as b. For a strictly convex preference relation, all the bundles on the line segment, excluding the end points, are better than b. See Figures 4.4a and 4.4b.

Example 4.7: Convexity of lexicographic preferences

Lexicographic preferences are convex by the following argument. Assume $(a_1, a_2) \succsim (b_1, b_2)$. If $a_1 > b_1$ then for every $\lambda \in (0,1)$ we have $\lambda a_1 + (1-\lambda)b_1 > b_1$ and thus $\lambda a + (1-\lambda)b \succ b$. If $a_1 = b_1$ then $\lambda a_1 + (1-\lambda)b_1 = b_1$. In this case $a_2 \geq b_2$ and hence $\lambda a_2 + (1-\lambda)b_2 \geq b_2$, so that $\lambda a + (1-\lambda)b \succsim b$.

Proposition 4.2: Characterization of convex preference relation

The preference relation $\succsim$ on $\mathbb{R}^2_+$ is convex if and only if for all $x^* \in X$ the set $\{x \in X : x \succsim x^*\}$ (containing all bundles at least as good as x^*) is convex.

Proof

Assume that $\succsim$ is convex. Let $a, b \in \{x \in X : x \succsim x^*\}$. Without loss of generality assume that $a \succsim b$. Then for $\lambda \in (0,1)$, by the convexity of $\succsim$ we have $\lambda a + (1-\lambda)b \succsim b$ and by its transitivity we have $\lambda a + (1-\lambda)b \succsim x^*$, so that $\lambda a + (1-\lambda)b \in \{x : x \succsim x^*\}$. Thus this set is convex.

Now assume that $\{x \in X : x \succsim x^*\}$ is convex for all $x^* \in X$. If $a \succsim b$ then we have $a \in \{x \in X : x \succsim b\}$. Given that b is also in $\{x \in X : x \succsim b\}$, the convexity of this set implies that $\lambda a + (1-\lambda)b$ is in the set. Thus $\lambda a + (1-\lambda)b \succsim b$.

The next result involves the notion of a concave function. A function $u : X \to \mathbb{R}$ is concave if for all $a, b \in X$, $u(\lambda a + (1-\lambda)b) \geq \lambda u(a) + (1-\lambda)u(b)$ for all $\lambda \in (0,1)$.

Proposition 4.3: Preferences with concave representation are convex

A preference relation on $\mathbb{R}^2_+$ that is represented by a concave function is convex.

Proof

Let $\succsim$ be a preference relation that is represented by a concave function u. Assume that $a \succsim b$, so that $u(a) \geq u(b)$. By the concavity of u,

$$u(\lambda a + (1-\lambda)b) \geq \lambda u(a) + (1-\lambda)u(b) \geq u(b).$$

Thus $\lambda a + (1-\lambda)b \succsim b$, so that $\succsim$ is convex.

Note that convex preferences may be represented also by utility functions that are not concave. For example, the convex preference relation represented by the concave function $\min\{x_1, x_2\}$ is represented also by the function $(\min\{x_1, x_2\})^2$, which is not concave.

The convexity of a strongly monotone preference relation is connected with the property known as decreasing marginal rate of substitution. Consider three bundles $a = (10, 10)$, $b = (11, 10 - \beta)$, and $c = (12, 10 - \beta - \gamma)$ for which $a \sim b \sim c$. When the amount of good 1 increases from 10 to 11, the consumer is kept indifferent by reducing the amount of good 2 by β, and when the amount of good 1 increases by another unit, he is kept indifferent by further reducing the amount of good 2 by γ. We now argue that if the consumer's preference relation is strongly monotone and convex then $\beta \geq \gamma$. That is, the rate at which good 2 is substituted for good 1 decreases as the amount of good 1 increases. Assume to the contrary that $\beta < \gamma$. Then $\beta < \frac{1}{2}(\beta + \gamma)$, so that by strong monotonicity $(11, 10 - \frac{1}{2}(\beta + \gamma)) \prec b = (11, 10 - \beta)$. But $(11, 10 - \frac{1}{2}(\beta + \gamma)) = \frac{1}{2}a + \frac{1}{2}c$, and the convexity of the preferences implies that $\frac{1}{2}a + \frac{1}{2}c \succsim c$, so that $(11, 10 - \frac{1}{2}(\beta + \gamma)) \succsim c \sim b$, a contradiction.

4.6 Differentiability

Consumers' preferences are commonly assumed to have smooth indifference sets, like the one in Figure 4.5a. The indifference set in Figure 4.5b, by contrast, is not smooth. A formal property of a preference relation that ensures the smoothness of indifference sets is differentiability. We define this property only for monotone and convex preference relations.

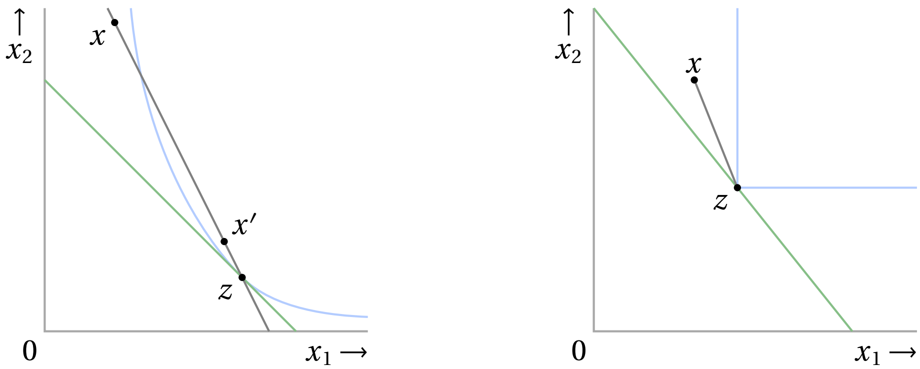
Definition 4.6: Differentiable preference relation

A monotone and convex preference relation $\succsim$ on $\mathbb{R}^2_+$ is *differentiable* if for every bundle z there is a pair $(v_1(z), v_2(z)) \neq (0, 0)$ of nonnegative numbers, called the consumer's *local valuations at* z, such that for all numbers δ_1 and δ_2,

$$v_1(z)\delta_1 + v_2(z)\delta_2 > 0 \quad \Leftrightarrow \quad \text{there exists } \varepsilon > 0 \text{ such that } z + (\varepsilon\delta_1, \varepsilon\delta_2) \succ z.$$

Geometrically, this definition says that for any given bundle z there is a line (like the green one in Figure 4.5a) such that (*i*) for any bundle x above the line, every bundle sufficiently close to z on the line segment from z to x (like x' in Figure 4.5a) is preferred to z and (*ii*) any bundle that is preferred to z is above the line.

The numbers $v_1(z)$ and $v_2(z)$ can be interpreted as the consumer's valuations of small changes in the amounts of the goods he consumes away from z. If his



(a) An indifference set for a differentiable preference relation.

(b) An indifference set for a preference relation that is not differentiable.

Figure 4.5

preference relation is differentiable, then for $\varepsilon > 0$ small enough the change from the bundle z to the bundle $z' = (z_1 + \varepsilon\delta_1, z_2 + \varepsilon\delta_2)$ is an improvement for the consumer whenever $v_1(z)\delta_1 + v_2(z)\delta_2 > 0$. (Note that only the ratio $v_1(z)/v_2(z)$ matters; if $(v_1(z), v_2(z))$ is a pair of local valuations, then so is $(\alpha v_1(z), \alpha v_2(z))$ for any number $\alpha > 0$.)

Figure 4.5b gives an example of an indifference set for preferences that are not differentiable. For every line (like the green one) through z such that all bundles preferred to z lie above the line, there are bundles (like x in the figure) such that *no* bundle on the line segment from x to z is preferred to z.

Lexicographic preferences are not differentiable. Suppose that the quantity of the first good has first priority and that of the second good has second priority. For any bundle z, the only vector $(v_1(z), v_2(z))$ such that for all δ_1 and δ_2 the left-hand side of the equivalence in Definition 4.6 implies the right-hand side is $(1,0)$ (or a positive multiple of $(1,0)$). However, for this vector the right-hand side of the equivalence does not imply the left-hand side: for $(\delta_1, \delta_2) = (0,1)$ we have $1 \cdot \delta_1 + 0 \cdot \delta_2 = 0$ although $(z_1 + \varepsilon\delta_1, z_2 + \varepsilon\delta_2) \succ (z_1, z_2)$ for $\varepsilon > 0$.

The following result, a proof of which is beyond the scope of the book, says that a preference relation represented by a utility function with continuous partial derivatives is differentiable and its pair of partial derivatives is one pair of local valuations.

Proposition 4.4: Local valuations and partial derivatives

If a preference relation on $\mathbb{R}^2_+$ is monotone and convex and is represented by a utility function u that has continuous partial derivatives, then it is differentiable and for any bundle z one pair of local valuations is the pair of partial derivatives of u at z.

Thus, for example, the preference relation represented by the utility function u defined by $u(x_1, x_2) = x_1 x_2$ is differentiable and for any bundle z, $(v_1(z), v_2(z)) = (z_2, z_1)$ is a pair of local valuations.

Problems

1. *Three examples.* Describe each of the following three preference relations formally, giving a utility function that represents the preferences wherever possible, draw some representative indifference sets, and determine whether the preferences are monotone, continuous, and convex.

   a. The consumer prefers the bundle (x_1, x_2) to the bundle (y_1, y_2) if and only if (x_1, x_2) is further from $(0, 0)$ than is (y_1, y_2), where the distance between the (z_1, z_2) and (z_1', z_2') is $\sqrt{(z_1 - z_1')^2 + (z_2 - z_2')^2}$.

   b. The consumer prefers any balanced bundle, containing the same amount of each good, to any unbalanced bundle. Between balanced bundles, he prefers the one with the largest quantities. Between unbalanced bundles, he prefers the bundle with the largest quantity of good 2.

   c. The consumer cares first about the sum of the amounts of the goods; if the sum is the same in two bundles, he prefers the bundle with more of good 1.

2. *Three more examples.* For the preference relation represented by each of the following utility functions, draw some representative indifference sets and determine (without providing a complete proof) whether the preference relation is monotone, continuous, and convex.

   a. $\max\{x_1, x_2\}$

   b. $x_1 - x_2$

   c. $\log(x_1 + 1) + \log(x_2 + 1)$

3. *Continuous preferences.* The preference relation $\succsim$ is monotone and continuous and is thus represented by a utility function u that is increasing and continuous. Show that for every bundle x there is a bundle y with $y_1 = y_2$ such that $y \sim x$.

4. *Quasilinear preferences.* A preference relation is represented by a utility function of the form $u(x_1, x_2) = x_2 + g(x_1)$, where g is a continuous increasing function.

   a. How does each indifference set for this preference relation relate geometrically to the other indifference sets?

b. Show that if g is concave then the preference relation is convex.

5. *Maxmin preferences.* Prove that the preference relation represented by the utility function $\min\{x_1, x_2\}$ is convex.

6. *Ideal bundle.* Show that the preference relation in Example 4.5, in which the consumer has in mind an ideal bundle, is continuous and convex.

7. *One preference relatively favors one good more than another*. We say that the preference relation $\succsim_A$ favors good 1 more than does $\succsim_B$ if for all positive numbers α and β we have

$$(x_1 - \alpha, x_2 + \beta) \succsim_A (x_1, x_2) \quad \Rightarrow \quad (x_1 - \alpha, x_2 + \beta) \succ_B (x_1, x_2).$$

a. Illustrate by two collections of indifference sets the configuration in which $\succsim_A$ favors good 1 more than does $\succsim_B$.

b. Explain why the preference relation $\succsim_A$ represented by $2x_1 + x_2$ favors good 1 more than does the preference relation $\succsim_B$ represented by $x_1 + x_2$.

c. Explain why a lexicographic preference relation (Example 4.6) favors good 1 more than does any strongly monotone preference relation.

Notes

The result mentioned at the end of Section 4.4 that every continuous preference relation can be represented by a continuous utility function is due to Debreu (1954). The exposition of the chapter, and in particular the presentation of differentiability, draws upon Rubinstein (2006a, Lecture 4).