Studies on Mathematics Education and Society

BREAKING IMAGES

ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

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1. Beginning

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The contributions in this book focus on critically analysing the relationship between mathematics as a discipline and mathematics as a school subject. The discontents of school mathematics are universally acknowledged and include questions such as: Why do so many people, however intelligent and successful, have feelings of inadequacy and alienation towards the subject? Why does mathematics education in school not seem to improve despite all the effort put into it? Our collective attempt to address such questions through radical rethinking begins by arguing that it is more productive to speak in terms of doing mathematics*, in a variety of senses, rather than using words that imply that mathematics* exists as some kind of entity. *In particular, we reject the notion of mathematics being independent of human agency. Such a reformulation is in line with recent developments in mathematics and the philosophy of mathematics that problematise the quest for a definitive and timeless definition of mathematics. Related developments in history of mathematics, anthropology, and related fields make it imperative to acknowledge historical, cultural, social, ethical, and political – in short, human – dimensions of mathematics and mathematics education. Multiple important themes that are generated by this perspective are summarised.*

The purpose of this book is to examine, critically and in their full complexity, relationships between conceptions of mathematics (mainly presented in Part 1 of this book) and the teaching/learning of mathematics in schools (mainly presented in Part 2 of this book).

The reader of this introductory chapter, and of the book as a whole, can hardly fail to become aware of the tension produced by the attempt to keep within reasonable length a discussion that involves negotiating a minefield of exploding concepts while trying to avoid omission of

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essential aspects. We have made certain decisions necessary to keep the scope within manageable bounds, such as essentially limiting the contexts to those of 'the West'. Thus, we do not address, for example, Indian philosophies of mathematics, or mathematics education in China. Discussion on mathematics education relates predominantly to that which happens in schools, as opposed to university mathematics education and learning in out-of-school contexts. The following sections outline some of the main themes of the book.

Conceptions of mathematics

In his book *What is Mathematics, Really?*, Reuben Hersh makes the following observation:

> The working mathematician is a Platonist on weekdays, a formalist on weekends. On weekdays, when doing mathematics, he's a Platonist, convinced he's dealing with an objective reality whose properties he's trying to determine. On weekends, if challenged to give a philosophical account of the reality, it's easiest to pretend he doesn't believe it. He plays formalist, and pretends mathematics is a meaningless game. (Hersh, 1997, p. 39)

We refer to such a formulation as a working philosophy of mathematics. It need not be well articulated, and, as indicated by Hersh, it need not even be consistent. Formal philosophies of mathematics have been elaborated in all directions (for overviews, see Benacerraf & Putnam, 1964; Hacking, 2014; Shapiro, 2000). As a term avoiding a sharp distinction between the two, we tend to refer to 'conceptions of mathematics'.

'What is mathematics?'

A short sampling of answers:

> Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true. (Russell, 1901, p. 1)

> Mathematics is the art of giving the same name to different things. (Attributed to Poincaré)

> Mathematics is the language in which God has written the universe. (Attributed to Galileo)

> Mathematics is the study of all possible patterns. (Sawyer, 1955, p. 12)

Answers to the question 'What is Mathematics?' may be enigmatic, aphoristic, religious, hubristic, aesthetic. However, one also finds answers that have been elaborated through deep philosophical investigations. Let us briefly recapitulate the positions of logicism, formalism, and intuitionism.

Since Antiquity, Platonism has been carefully articulated and further developed. Gottlob Frege (e.g., 1967) reworked Platonism in relation to mathematics into a completely new format, claiming that the idealised and permanent mathematical objects are sets. In this way, he launched the logicist programme, which tries to show that mathematical entities in fact are logical entities, and that mathematical statements are logical statements. In *Principia Mathematica*, Alfred Whitehead and Bertrand Russell (1910–1913) elaborated this programme to the extreme. As already quoted, Russell characterised mathematics as a subject in which 'we never know what we are talking about'. In a more serious mood, he declared George Boole's *Laws of Thought* to be about formal logic, adding 'and this is the same thing as mathematics' (Russell, 1901, p. 1).

David Hilbert, wanting to systematically address mathematical theories, advocated formalising mathematics so that they could be investigated with respect to, for instance, consistency and completeness. This programme led directly to the idea that mathematics can be identified with formal systems. In *Outlines of a Formalist Philosophy of Mathematics*, Haskell Curry (1970) provides a comprehensive presentation of formalism and what it means to identify mathematics with formalism.

A third answer to 'What is mathematics?' comes from L. E. J. Brouwer (e.g., 1913), who formulated intuitionism as a philosophy of mathematics. According to Brouwer, formalism represents a complete misunderstanding of mathematics and formal structures. While formalists see formal structures as being precise expressions of mathematics, intuitionists view formal structures as imprecise and, in many cases, inappropriate approximations to mathematics. Wagner (2017) summarises intuitionism as questioning 'any mathematics that cold not be finitely constructed starting with counting a sequence of moments (in a Kant-like framework of temporality)' (p. 17). In this way, Brouwer characterised mathematics as a human, mental, activity.

Philosophical answers to the question 'What is mathematics?' reveal multiple conceptions of mathematics. Some see mathematics as an essential constituent of our world while others consider it as man-made. Such diversity and contrasts suggest that a search for a definitive characterisation of the *essence* of mathematics is a chimera, albeit one that, like the quest for the Philosopher's Stone, stimulates productive inquiry.

Posing a better question

We suggest that 'What *is* mathematics?' is not a good question. 'Mathematics' means a lot of different things for school students, for engineers, for philosophers in contemporary times, in the late nineteenth century, in Antiquity. The very grammar of the question tempts us to search for a universal essence of mathematics. However, how such an essence could be found and verified constitutes an unsolved, arguably unsolvable, philosophical problem. Every attempt to capture the essence of mathematics entails the danger of generalising a particular perspective at the expense of others. Mathematics has, and will continue to evolve, a history, and the families of activity systems that involve mathematics are diverse.

Semantically speaking, in terms of the discourse theory of Ernesto Laclau and Chantal Mouffe (1985/2001), 'mathematics' constitutes a floating signifier, a concept whose strength in combining with other concepts, activities, and expectations depends on its conceptual flexibility, on its openness to assume different facets of meaning in different discourses. It is also worth questioning to what extent 'mathematics' is best regarded as a noun. Could it be interpreted more like a verb? In fact, we are going to suggest a shift to thinking about *what can be done through mathematics*. We contend that the characterisation of mathematics is more usefully framed not in terms of an entity, but in terms of what humans, individually and collectively, *do* when they engage with mathematics.

Hans Freudenthal, inspired by Brouwer and by intuitionism, highlighted again and again that 'mathematics is a human activity'

(Chapters 7 and 8, this volume). Let us clarify what we mean by 'human activity'. For emphasis, we may instead use the phrase 'social activity', reflecting an orientation that envisages collective mental activity, not just what Brouwer took to be an activity of a single mind. Such an individualistic formulation may seem natural when we consider a child having a breakthrough insight, or a solitary mathematician struggling with a proof. Even in such circumstances, however, the social nexus is still there. The mathematician is part of a community with well-established norms (Chapter 9, this volume) that has worked on the problem; the child is in an educational setting. Indeed, to the extent that thought may be considered internal communication (a question we will not attempt to address), it is inherently socially grounded, in particular linguistically.

School and its associated practices (and not just learning and teaching) constitute a very particular form of historically evolved social activity. In other cultural settings, there are very different forms of learning and teaching, including those in which 'doing' and 'learning' are embedded in the same activity. Contrast that with the familiar answer from mathematics teachers to the question 'Why are we doing this?', namely some variant of 'Because it will be useful to you later'.

For all of these reasons, we contend that a better question than 'What is mathematics?' is to ask something like 'What do people do when they use mathematics within an activity system?'. Such a shift away from essentialist to performative paradigms is not unique to the philosophy of mathematics. For example, there is a parallel with Ludwig Wittgenstein's (1997/1953) later work of interpreting language through its use in what John Searle (1969) called 'speech acts'. The conception of language shifted from a descriptive perspective to a performative perspective. In a similar way, then, we want to pay particular attention to performative features of mathematics, which are highlighted by Ole Ravn and Ole Skovsmose (2019) through their formulation of a four-dimensional philosophy of mathematics. We may point to ethnographic studies of people doing mathematics in workplace contexts, for example. George Pólya's (e.g., 1962) emphasis on how mathematicians behave stems from a similar motivation.

Seeing mathematics as a social activity has profound implications. It shifts the balance away from 'mathematics' as something that *exists* (in whatever sense) to something that is *done by people*. It makes it natural to adopt both the historical and diversity lenses and prompts many other considerations that are relevant to both mathematics as an academic discipline and mathematics education. It becomes natural to consider how conceptions of mathematics have changed over historical time and to acknowledge that differently situated people might mean different activities when they refer to mathematics, even one and the same person might refer to different activities. For example, when we refer to mathematics in academic situations, activities such as defining concepts, testing hypotheses, and formulating proofs are central activities, but often they are not typical activities in school mathematics.

To signal and emphasise that one aspect among many is being highlighted, we use 'as' rather than 'is' in phrases such as 'Mathematics as a process of discovery' (see Ravn & Skovsmose, 2019). So, in addition to mathematics as academic discipline, we will also talk about, for example, mathematics as cultural constructions, mathematics as practices in work, mathematics as engineering techniques, mathematics as school subject, and so on. We allow ourselves to be unsystematic in our use of 'mathematics as...' and we fully recognise that we have to cope with a fuzzy way of using the words. Clearly, this phrasing in terms of families of practices in which mathematics is embedded is closely aligned with the concept of Ethnomathematics (Chapters 10 and 17, this volume).

Evolution of academic mathematics

As a human activity, that set of practices that we term 'academic mathematics' has a long history (Chapter 2, this volume). In the course of that history, radical conceptual restructuring has taken place, and continues to take place. To use the most familiar example, what is meant by 'number' stretches from the 'natural numbers' 1, 2, 3, ... to the equation $e^{i\pi} = -1$ and beyond.

A first central question regarding such developments is: What are the processes through which conceptual restructuring occurs? Answers to this question minimally include the following:

- In response to human needs. For example, because of its late development, we have a relatively clear historical picture of how probability theory was initially motivated by the needs of gamblers, and developed in close proximity to situations such as jury trials, risk assessment, social theories of the nature of man, and so on (Hacking, 1990).

- By asking 'What if…?' questions, such as 'What if we don't assume Euclid's fifth axiom?', a question that led to revolutionary developments in geometry.
- Through the symbiotic development of tools, including representational tools, for example, coordinate geometry based on the Cartesian representation.
- Through making connections between apparently disjoint fields, notably the translatability between geometry and algebra achieved by René Descartes (discussed at length by Hacking, 2014).
- Through internal crises, disequilibria, a famous example being the realisation that the diagonal of a square is incommensurable with its side.
- Through the detachment of mathematical structures from their origins in systematised situations. A clear example is the concept of 'group' which eventually came to be defined as a set, together with an operation on ordered pairs thereof, having certain properties. Given this definition, mathematicians could pursue their researches independently of any particular examples or applications of group structures.
- Through the reconceptualisation of conceptual entities within mathematics. The case study by Imre Lakatos (1976) on a theorem about polygons is a prime example; changed ideas of the nature of mathematical proof given the advent of computers is another.

A second key question is 'To what extent is the development of mathematics necessary, and to what extent contingent?' Rafael Núñez (2000) argues that it is not a binary choice, stating that that 'mathematics is not transcendentally objective, but it is not arbitrary either (not the result of pure social conventions)' (p. 3). There are mathematical developments that feel like they could not have happened otherwise

– for example, the extension from natural numbers to rational numbers and directed numbers. It is not so obvious, however, when it comes to the question posed by Núñez: 'Have you ever thought why (I mean, really *why*) the multiplication of two negative numbers yields a positive one?' (p. 3)

That the development of academic mathematics proceeds in a way that is absolutely predetermined is arguably disproved by the diversity within it. For example, Raju (2007, p. 413) declared that within European mathematics there are two streams:

1. from Greece and Egypt a mathematics that was spiritual, anti-empirical, proof-oriented, and explicitly religious, and
2. from India via Islamic countries a mathematics that was pro-empirical, and calculation-oriented, with practical objectives.

Raju's (2007) work is also an important contribution to one aspect of Ethnomathematics, namely the construction of a counter-narrative to the myth that academic mathematics is a purely European achievement.

Is doing mathematics inherently beneficial to humankind?

In the European context, since Antiquity, mathematics has been admired and celebrated, while, in academia, a critical conception of mathematics has only been articulated within the last century. Plato admired mathematics, which showed what it could mean to enter the world of ideas. Via the human senses such access was not possible, but through rationality, it was assumed, we can explore properties of idealised objects. The Platonist admiration of mathematics turned into a celebration of Euclid's *Elements*, which brought together an axiomatisation of geometry that right up to the late nineteenth century was considered to be perfect, serving as the epitome of the systematisation of mathematics within formal structures, and taken as the role model for how to build theories in science.

The admiration of mathematics acquired more fuel through the so-called scientific revolution. The people contributing to this were deep believers in God, as, for instance, Isaac Newton. They saw the world as created by God, meaning that insight and understanding of nature meant insight and understanding of God's creation. God had inserted laws of nature that could be captured by mathematics, truly an overwhelming insight. Through mathematics we human beings become able to grasp the rationality of God! When the natural sciences, following a protracted ideological struggle, separated from religious beliefs, the celebration of mathematics continued, and mathematics became nominated as the language of science. The celebration of mathematics has also become an integral part of much philosophy of science (e.g., Shapiro, 2000).

In contemporary circumstances, practitioners and proponents of mathematics (more generally the fashionable complex of Science, Technology, Engineering, and Mathematics, STEM) enjoy a great deal of political and cultural capital. In political and economic media discourse, statements to the effect that high achievement in STEM education is essential for economic competitiveness in the global marketplace are pervasive. A preponderance of what is written or spoken about mathematics in public, political, and academic discourses reflects an unexamined belief in what Paola Valero (2004) called 'the unquestioned intrinsic goodness of both mathematics and mathematics education [that represents] the core of its "political" value' (p. 13).

In this book, we leave behind the blind admiration of mathematics and consider the emergence of a critical stance towards mathematics, in particular its dehumanising effects (Chapter 5, this volume). The most concerted critique has emanated from within the group of critical mathematics educators (Chapter 11, this volume; and see Greer & Skovsmose, 2012, for a history of that movement). Relatively few mathematicians have expressed a critical attitude towards what people have done using mathematics. Writers commenting on the human condition who have done so include, notably, Charles Dickens, who was repelled by the class oppression that was exacerbated by the Industrial Revolution (Chapter 12, this volume).

Perhaps we should make clear that we by no means discount the very many ways in which mathematics has been, and can be, used to benefit our lives both practically and intellectually. However, given that there is no lack of writing in praise of mathematics, we feel the need to emphasise rather its problematic uses, including in the service of imperialism, for advancing the techniques of war, and its inextricable links with capitalism.

Ubiratan D'Ambrosio concluded his paper introducing Ethnomathematics as follows:

> Ideology [...] takes a more subtle and damaging turn, with even longer and more disrupting effects, when built into the formation of the cadres and intellectual classes of former colonies, which constitute the majority of so-called Third World countries. We should not forget that colonialism grew together in a symbiotic relationship with modern science, in particular with mathematics, and technology. (D'Ambrosio, 1985, p. 47)

Beyond the material military contributions to colonial conquest through technology, we have to consider the symbolic violence of suppressing other forms of knowledge and replacing them with European epistemologies and practices.

Mathematics has long been used in the service of war, and many mathematicians have devoted their talents to the design of more effective ways of killing people. Others have used mathematics for the more efficient management of warfare. A very strong statement was made by Zygmunt Bauman (1989) that the Holocaust was not an anomaly within modernity but, in its monstrous effectivity, depended on the most modern practices of organisation, including mathematics (Chapter 5, this volume).

Again, the use of mathematics in the service of capitalism constitutes a vast subject and here we merely draw attention to some specific aspects. Most fundamental, perhaps, are the connections between the great abstractions of number and capital, intermediated through money as represented materially and, increasingly, in virtual forms. Economic and political theorists can present various dynamic system analyses of the possibly irreversible development of the particular pathological form of capitalism currently in the United States and beyond. We may consider to what extent contemporary mathematics education within particular political regimes plays a role in preparing children to be active proponents or passive citizens within capitalist systems.

In view of the discussion above, we take the position that it is no longer possible for mathematicians (or scientists, or any scholars) to claim ethical/political neutrality, such a claim in itself being a kind of ideology (Chapter 4, this volume). Specifically, in Chapter 3, Skovsmose discusses the views of G. H. Hardy as presented in *A Mathematician's Apology*, in which Hardy (1967) suggests that a mathematician can operate as a pure intellectual, with no responsibility for what is done with her/his work. Another mathematician, Chandler Davis (2015) issued a different kind of apology – not in the sense of 'apologia' – when he regretted that he and other mathematicians had not done more to oppose war, including the Mutually Assured Destruction (MAD) principle that guided policy during the Cold War, and was substantially based on the work of John von Neumann and others on game theory.

In direct opposition to Hardy's stance, Ubiratan D'Ambrosio, in the manifesto for 'Non-killing Mathematics',[1] asserts that it is not enough for mathematicians to do good work, they must pay attention to what will be done using that work, and that it is not enough for mathematics educators to teach students well, they must pay attention to what those students will do with what they have been taught.

Development of mathematical understanding under instruction

In considering relationships between the development of mathematics by humankind and the development of mathematical knowledge and understanding in a contemporary student, the most obvious point is that the former occurred over millenia as opposed to a small number of years. A child today is expected to deal, at least procedurally, with mathematical content that historically took multiple good brains collectively a very long time to figure out.

As stated by Freudenthal (1991), 'we know nearly nothing about how thinking develops in individuals, but we can learn a great deal from the development of mankind' (p. 48). In response to his own question as to whether the learner should repeat the learning process of mankind, his response is 'of course not'. Instead, his recommendation is that 'the learner should reinvent mathematizing rather than mathematics; abstracting rather than abstractions; schematizing rather than schemes; formalising rather than formulas; alogrithmising rather than algorithms; verbalising rather than language', which chimes with

1 Ethics/Nonkilling/Mathematics (2024, April 5). *Wikiversity*, https://en.wikiversity.org/wiki/Ethics/Nonkilling/Mathematics

our emphasis on actions. In the same spirit, Pólya argued for children having the opportunity to experience problem-solving for themselves: 'How can you know if you like raspberry pie if you have never tasted it?' (Pólya, 1945, p. v).

Another important point has been clearly stated thus:

> Teaching is one of the immense social influences that can affect a child, but its effects can be out of proportion to any other kind of social influence once the first beginnings of a child's life are past. In it once again knowledge builds on knowledge, but the form of experience that makes it possible is really quite unlike those forms of experience that come the individual's way when teaching is not involved. (Hamlyn, 1978, p. 144)

Multidiversity

In terms of mathematics as a human activity, 'multidiversity' relates to differences among and within families of mathematical activities emergent from their cultural and historical underpinnings, including forms of life, worldviews, cognition, language, value systems, and so on. In terms of school mathematics, it relates to the myriad of differences, interacting in complex ways, among students (and also among teachers, a story in itself). These include, notably, ethnic diversity (Chapter 18, this volume) and gender (Chapter 19, this volume). Within mathematics education, much of the foundational work addressing diversity has been concerned with 'equity' and 'access'. The sloganising of these terms demands more careful analysis (e.g., Martin, 2019; Pais, 2012) and we pinpoint the following preliminary questions and comments:

- Access to what? Many if not most of the exhortations to improve access takes mathematics-as-school-subject as an unexamined given.
- Equity on whose terms? Is it merely assimilation, involving the denial of cultural identity?
- Beyond equity and access lie identity and agency.

All of these, of course, are intensely political in nature.

In current circumstances, we can observe a hegemonical struggle between acknowledgment and valorisation of diversity in all its aspects, and multifaceted forces tending towards homogenisation, linked with globalisation (Westernisation), corporatisation, metrification, and so on. Such homogenisation is certainly prominent within mathematics education. Perhaps the most obvious manifestation is in curricular documents, for which the Common Core State Standards within the United States may serve as an example. We draw attention to its stated principle of benchmarking with similar projects from other countries, contributing to a process of convergence towards global uniformity, exacerbated by the effects of the international comparison industry (Chapters 15 and 16, this volume).

Parenthetically, as a parallel, think of the onward march of English as a global language, among the consequences of which is a significant distortion of our field. This book, in English, has been written by speakers of many languages and edited by two people for whom English is a foreign language and one who grew up speaking English because of early colonisation and linguicide.[2]

Epistemological pluralism is another central issue, including from the perspective of mathematics-as-discipline. Rik Pinxten, Ingrid van Dooren, and Frank Harvey (1983), who studied the fundamentally different epistemology of the Navajo people, in particular in relation to space, commented that:

> Through a systematic superimposition of the world view and thought system of the West on traditional non-Western systems of thought and action all over the world, a tremendous uniformization is taking hold [...] The risks we take on a worldwide scale, and the impoverishment we witness is – evolutionarily speaking – quite frightening. (pp. 174–175)

As a closing comment, we observe that in terms of families of mathematical practices, there is obvious diversity within mathematics as cultural constructions, mathematics in work practices, mathematics of everyday life, and, indeed, within academic mathematics (e.g., Hersh, 2006). Yet this diversity is not generally manifest in school mathematics; we regard that as a problem.

2 To respect authors' linguistic preferences and cultural identities, authors of each chapter have opted to follow British or American English in spelling and punctuation.

Mathematics education as a research field

A survey volume edited by Anna Sierpinska and Jeremy Kilpatrick (1998) is tellingly titled *Mathematics Education as a Research Domain: The Search for Identity*. The emergence and development of mathematics education as a field has seen a diversification of influential disciplines and methodologies – broadly speaking, the balancing of technical disciplines by human disciplines such as sociology and anthropology, and formal statistical methods by interpretative methods of research and analysis.

The desire to have clearcut methodologies avoiding complex human judgments has passed through many manifestations from the early alignment with logical positivism and related positions. In his address to the first International Conference on Mathematics Education in 1969, Edward Begle explicitly recommended the empirical-scientific approach through a program of identifying the important variables and systematically studying the relations between them. In Begle (1979), he confessed to feeling depressed that a decade of experimental work had produced little progress. In fact, a range of theoretical frameworks may be characterised as attempts to apply scientific precision to the complexity of understanding and improving mathematics education – behaviourism, information-processing theory, Artificial Intelligence, neurocognition – aligned with a reliance on narrowly defined standards of empirical research and statistical modelling. Kilpatrick (1981), in a paper entitled 'The Reasonable Ineffectiveness of Research in Mathematics Education', cited Irving Kristol (1973), who raised the question why we can send a man to the moon, but cannot improve mathematics education, and answered it by pointing out that the former is a technical problem, the latter is a human problem.

Academic mathematicians' claims over mathematics education

The most obvious difference between mathematics-as-discipline and mathematics-as-school-subject lies in the nature of the populations involved. Picture a pyramid representing all those who are taught mathematics in school. A very small peak corresponds to those who will become academic mathematicians. A rather larger zone beneath that corresponds to those, such as engineers, that will use significant technical mathematics. The largest part of the pyramid represents people who secure material support for those at the peak and who do, indeed, use mathematics, but most often learned in context as needed, using situated procedures unrelated to what they learned in school, and mediated by tools (Lave, 1988).

Accordingly, we ask 'To what extent, and in what ways, should academic mathematicians be accorded control over school mathematics education?' Mathematicians have vested interests in the reproduction of their kind, and so may be suspected of bias, as well as developmental ignorance, by which we mean that, in their expertise, they forget what it is like to struggle with mathematics. We put forward two propositions for consideration. The first is that mathematicians should not dominate school mathematics – simply put, mathematics education is far too important to be left to mathematicians. The second is that mathematics education is about much, much more than the transmission of a subset of accumulated and systematised mathematical knowledge and techniques. We take issue with the position that the predominant role of those who work in mathematics education should be simply to study and implement better ways to effect this transmission. For a clear statement of that position, broadly speaking, see the book edited by Michael Fried and Tommy Dreyfus (2014).

The most obvious manifestation of mathematicians shaping mathematics education is through the formulation of curricula. The Common Core State Standards for Mathematics in the United States, mentioned above, may be taken as representative of the search for the perfect model. It was primarily designed by three mathematicians, albeit with an advisory group that included mathematics educators. But there are many, many other actors that have direct and indirect roles in shaping mathematics education in the United States, as analysed in great detail by Mark Wolfmeyer (2014).

We suggest that the uses of the term 'mathematics' in political discourse support an unreasonable sway over the policies and administration of mathematics education. Both reflecting and influencing what politicians do, the images of mathematics and mathematics education among the public in general (Chapter 20, this volume) matter greatly.

School mathematics as an instrument of the state

> The advancement and perfection of mathematics are immediately connected with the prosperity of the state. (Attributed to Napoleon, 1800)

> The functioning of the modern state presupposes a variety of mathematical technologies – accounting, statistics, and much more. Mathematics, on its part needs the institutions of the state (schools, universities, research institutions, etc.) to secure financing, recruitment and the rearing of competence. (Høyrup, 2019, p. 635)

Ian Hacking (1990) has documented, in painstaking detail, the ways in which the formal mathematics of probability and statistics developed within socio-political contexts, in close relationship to changing views of the nature of humans, and in the service of states. In a rare overtly political statement, he trenchantly observed that:

> We obtain data about a governed class whose deportment is offensive, and then attempt to alter what we guess are relevant conditions of that class in order to change the laws of statistics that the class obeys. (Hacking, 1990, p. 119)

The two most obvious mechanisms through which states control school mathematics are curriculum (in concert with mathematicians, see above) and standardised testing (in concert with psychometricians and others). We assert that curriculum, historically, has been characterised by inertia and stasis in terms of content and pedagogy, and as argued within this book, accords little weight to the needs of people in general.

Arguably, however, the sharpest tool for state control of school mathematics lies within the proliferation of standardised testing, locally, nationally and globally, within which mathematics has a particular importance. On the one hand, mathematics is implicated because it underpins the models used to construct such testing and interpretations of the results and, at a deeper level, the culture of affording unjustified authority to numbers (e.g., Porter, 1975) and mathematical models (O'Neil, 2016; Skovsmose, 2005). And the imposition of such testing constrains and distorts mathematics teaching and learning (for a detailed historical survey by a battle-scarred participant, see Chapter 14, this volume).

Formative assessment, in the sense of assessment by a teacher in the course of interactions with students, forms an integral part of learning and teaching within a long-term relationship. Such a process has at least the potential of affording an effective form of communication. By contrast, summative assessment, in its typical forms, is a form of communication whose flaws are compounded across many stages (Miller-Jones & Greer, 2009). In the United States, the standard use of the term 'achievement gap', implying a deficit model, instead of 'differences in test scores' is another pernicious use of language. And accreditation in mathematics creates a barrier to educational and financial opportunities through imposing requirements unrelated to the actual needs of chosen career paths, as has been particularly well documented by Hacker (2016).

Turning to the escalating power of international comparative assessment exercises, Christine Keitel and Kilpatrick (1998) concluded a critique with the following damning assessment:

> The studies rest on the shakiest of foundations – they assume that the mantel of science can cover all weaknesses in design, incongruous data and errors of interpretation. They not only compare the incomparable, they rationalize the irrational. (p. 254)

In their edited volume, *Education by the Numbers and the Making of Society*, Sverker Lindblad, Daniel Pettersson, and Thomas Popkewitz (2018) analyse the dominance of international educational assessments (in which mathematics has a pre-eminent place in terms of its role in constructing models and in terms of its prominence as subject-matter of tests) in shaping educational policymaking on a global scale, to the extreme of shaping the right kind of people and the right kind of countries. Most fundamentally, they present arguments about the harmful effects of uncritical obeisance to the authority of numbers, and about the use of statistical and modelling techniques in furthering the rise of neoliberal hegemony in education.

While curriculum and testing are the most blatant instruments, there are more subtle ways in which mathematics education may both reflect and frame forms of life and worldviews. Here we exemplify core elements of the standard school mathematics curriculum and their possible effects:

- In many systems of mathematics education, considerable emphasis is given to procedural fluency with algorithms. Might it be that this helps form a disposition for following rules, and abdicating responsibility for making personal judgments? (Skovsmose, 1994).
- It has been amply documented (e.g., Verschaffel, Greer, & De Corte, 2000) that children manifest suspension of sense-making when solving word (or story) problems in mathematics. Is it going too far to suggest that this kind of experience over years of schooling contributes to inculcating a frame of mind whereby a person uncritically accepts an unproblematic mapping of situations in the world onto equations? (see, e.g., Porter, 1975).
- More generally, it could be argued that the nature of mathematical modelling in general is poorly conveyed in mathematics education, failing to address a critical attitude to modelling that takes into account the motivations of the modellers, the limitations of representational and physical modelling tools available, the reliance of models on assumptions made, the difficulty of gauging the effects of simplification, the complexities of interpretation, and the nuances of communicating conclusions. Accordingly, mathematics education typically fails to prepare students to become citizens with a critical disposition and a desire to achieve and wield agency.
- A specific aspect of viewing the world that teachers and users of mathematics may unwittingly promote is the implicit rule that anything can be measured on a single dimension (Horkheimer & Adorno, 1944/1997). Once that is done, there are numerous implications, such as that averages can be worked out for different populations and compared (the history of measurements of intelligence provides an obvious example).

Final comments

Commentaries on mathematics education in schools – from students, parents, teachers, mathematics educators, researchers, politicians, and people in general – tend to be dominated by discontents and a sense of puzzlement about why such education seems to be unsuccessful in many ways despite the efforts put into improving it. In this book we argue that one starting point in addressing these discontents and their causes is a back-to-basics analysis of what is meant by 'doing mathematics', in particular by people designated as 'mathematicians', and how that vast diversity of activities contributes to shaping what happens in school classrooms. Throughout, we emphasise that the doing and teaching and learning of mathematics are situated in historical, cultural, social, and political – in short, human – contexts.

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