Studies on Mathematics Education and Society

BREAKING IMAGES

ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

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2. Why and how people develop mathematics

Brian Greer

The development of mathematics by humans has a long and unfinished history. In this, necessarily highly selective, overview, the discussion is framed in terms of the environments – physical, cultural, socio-political, specialised – within which people, including those designated as 'mathematicians' do what is called 'mathematics' in all its many forms. These forms include the traditional divide between 'pure' and 'applied'. A distinction is drawn between internal and external processes driving the development, and within internal drivers between those of creation and those of systematisation. The links between this chapter and Chapter 13 are stressed throughout.

Introduction

> Philosophers, like most other people who think about it at all, tend to take 'mathematics' for granted (Hacking, 2014, p. 41).

Arguably, Hacking's observation also holds true for most mathematicians, mathematics teachers, researchers on mathematics education – and everyone else. A major thrust of this book is to combat this tendency.

One of the most important and powerful antidotes to taking mathematics for granted is to examine the history of people – in particular the special kinds of people who are designated as 'mathematicians' – creating, chronicling, developing, systematising, applying what people call 'mathematics' or 'doing mathematics'.

A historian of mathematics faces the problem faced, *mutatis mutandis*, by the anthropologist, the child psychologist, the therapist, and many

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others, namely how to understand, from within one's one cultural and epistemological frameworks, those of the Other. As pointed out by scholars who have done the hard work, notably Jens Høyrup (Greer, 2021), some historians of mathematics address this challenge better than others.

In the context of China, but with general application, Christopher Cullen (2009) made a fundamental point in ruling out

> the idea that there is a priori a universal ahistorical, cross-cultural 'natural kind' called 'mathematics' that can simply be located and studied once one can penetrate the linguistic barrier to see what it is called in Chinese, and on which one can simply impose all the structures and expectations that a modern person finds in the subject called 'mathematics' in twenty-first-century English. (p. 592)

And, as with all history, the history of mathematics is complexified by gaps, errors of translation and interpretation, ideologically motivated falsifications, and other imperfections in the record. As a particularly striking instance, if you, the reader, would agree with the statement 'Pythagoras was a mathematician', you are recommended to read the entry on Pythagoras in the *Stanford Encyclopaedia of Philosophy*, available online (Huffman, 2018). And the entry on Socrates, in which it is stated that: 'Each age, each intellectual turn, produces a Socrates of its own' (Nails & Monoson, 2022).

It will be obvious that, in the service of writing this chapter, draconian selection was inevitable. The range of educational systems considered is limited. Topics are chosen with an eye to the arguments advanced in Chapter 13 in this volume. Thus, the preponderance of mathematical content addressed does not go beyond that of school mathematics. There is heavy reliance on what I judge to be load-bearing examples.

As a simple but convenient scheme, I frame the discussion by asking what are the 'drivers' of mathematical development, choosing that word to connote both impulsion and steering, the 'why' and the 'how' of the chapter title. I distinguish between *external* and *internal* drivers. The former are framed in terms of adaptations to environments – physical, cultural, political. A theme throughout is the relationship between the two faces of mathematics – on the one hand, the decontextualised codifications of accumulated mathematical knowledge and, on the other, the contextualised applications of mathematics to aspects of physical and human reality.

I further divide the discussion of internal drivers into those relating to acts of creating mathematics, and those relating to acts of systematising – again an obvious simplification that bears on discussion of a number of important issues, such as the fluid relationship between diversification and unification within mathematics, ways in which the development may be considered as following an inevitable trajectory or being contingent, and the relative contributions of individuals and collectives.

Internal drivers shape the discipline that emerged as a self-aware field of human activity in diverse milieux, with their own subcultures and norms, as 'constructed environments'. There are also special-purpose constructed environments relating to particular activity systems that mathematics can serve, such as military engineering.

The history of mathematics makes it abundantly clear that its development is a long and difficult process, and that constitutive of that development are epistemological crises and their resolutions. Periods of relatively steady elaboration and consolidation are punctuated by discontinuities.

In discussing mathematical creativity, I do not focus on the stories of individual triumphs that are often prominent in superficially 'popular' histories; instead, the emphasis is on collective aspects and on some of the salient factors conducive to the gaining and dissemination of new insights. In this respect it is difficult to overstate the importance of material representations, including the revolutionarily new resources made available through computer technology.

Turning to systematisation of mathematical knowledge, it is argued that while many aspects of the development of mathematics are contingent and subject to cultural diversity, that development is not arbitrary, since mathematics is an activity of humans existing in bodies, within social groupings, on a planet that affords underpinnings for mathematics, notably countable entities. Thus, any systematisation will reflect the balance between contingency and constraints. In particular, those constraints are manifest in general mechanisms of development, variously described in terms of hierarchical levels with each succeeding level building on its predecessor, well articulated by Hans Freudenthal

(1991), or Piagetian notions of successions of local equilibria – permeated throughout by the dialectical imperative. Along the same lines, the mathematician William Thurstone (1994) invoked recursion:

> As mathematics advances, we incorporate it into our thinking. As our thinking becomes more sophisticated, we generate new mathematical concepts and new mathematical structures: the subject matter of mathematics changes to reflect how we think. (p. 162)

Next, with a narrowing of focus to 'pure' or 'theoretical' mathematics, attention is given to the emphasis within 'modern mathematics' on elusive, temporary and local, aspirations for certainty such as absolutely precise definitions, irrefutable proofs, impeccable structures. A particular example examined in some detail is the Bourbaki enterprise that enjoyed considerable influence within academic mathematics for much of the twentieth century. That analysis illuminates tensions between mathematics as an academic discipline and mathematics as a school subject, and debate over the extent and nature of the influence of the former over the latter.

The chapter concludes with a brief summary and look ahead to Chapter 13.

External drivers

Mathematical practices may originate in the interactions between the human species and their physical environments, but humans, from a very early stage, have felt needs beyond the necessities of staying alive, including needs that may be described as spiritual, aesthetic, ludic, and the need for explanations and understanding. Thus, astronomy, which has been prominent for so long in so many cultures, has practical aspects relating to navigation, and has also been one of the salient areas for the metanotion that the physical world is governed by laws that can be mathematically framed, and it also has deeply religious connotations.

Then I briefly address the roles of mathematical activities within socio-political environments, with particular attention to how the discipline exists in a symbiotic relationship with the state, reflected in what might be called 'the unreasonable political effectiveness of "mathematics"' (where the quotation marks signal that what is being referenced is the propagandistic use of the word.)

Looking ahead to the next two sections, I consider some of the ways in which, as mathematics emerged as a recognised discipline with its acknowledged experts, creators, systematisers, practitioners, and teachers, external drivers have interacted with the drivers internal to the discipline. And, as an overarching theme, it is proposed that mathematics has 'two faces', one abstract and formal, the other relating to 'the real world' (a concept that I will not attempt to define, but assume to be meaningful in some way to the reader).

Physical environments, practical needs

Humans originally developed practices involving mathematics as part of adapting to their physical environments and the practicalities of survival. There are many experiences underpinning aspects of mathematics that are universal – birth and death, the force of gravity, cycles of day and night, seasons, and tides, observations of the night sky, objects and other entities that afford counting (fingers, prenatally listening to the maternal heartbeat), the approximate symmetry of the human body, and on and on. At this point in history, we should add finiteness in its multiple manifestations as an inherent aspect of the planet we inhabit.

As a counterpoint to universality (there is always a counterpoint) there is diversity in physical environments. It might be expected, for example, that the spatial epistemology, in interaction with visual perception, of people living in a dense forest would differ from that of people living on a treeless plain. One school of thought attributes diversity within the human race to climatic and environmental variation.

As already alluded to, a natural starting point is the human body, with obvious relevance to counting, measuring, perception, movement... The use of the vocal tract, mouth etc. for communication, evolving into language, was foundational for social development, and there followed the emergence of writing which enables, to a significantly greater extent that oral transmission, the extension of communication across space and time (Kaput & Schaffer, 2002). Writing also exemplifies the essentially human (though not exclusively so) characteristic of the use of tools extending the functionality of the body, underlying the emergence of cultural evolution beyond biological evolution.

Practices involving mathematics have been implicated in all forms of interactions of our species with the physical environment: adapting, observing and predicting, recording and organising data, understanding and explaining, controlling and changing to the point of destruction. Alan Bishop (1988) listed six families of practices significantly imbued with mathematical connotations that are found in essentially all cultures, namely counting, locating, measuring, playing, designing, and explaining. The first three, broadly speaking, represent ways of interacting with the physical environment in service of practical requirements, while the last three entail aspects that transcend, to a greater or lesser extent, the immediate needs of survival, as taken up in the next section.

Cultural environments, supra-utilitarian desires

As humans came to live within increasingly complex social/cultural environments, practices involving mathematical elements transcended issues of survival and day-to-day life. The study of mathematics may have been significantly motivated by contemplation of an immortal soul in the face of the ephemerality of bodily death. For many of the recognised greats of European mathematics, even into relatively recent times, the links to (broadly speaking) religious beliefs have been extremely strong (and often overlooked in histories that emphasise the rationality of the 'great men [sic] of mathematics'). Perhaps the hope of finding non-tautologous absolute certainty through mathematics in recent centuries is related to the loss, with the growth of scientific worldviews, of the feeling of absolute certainty attainable through blind religious faith.

Aesthetic impulses run deep. Franz Boas (1927/1955) concluded that:

> No people [...] however hard their lives may be, spend all their time, all their energies in the acquisition of food and shelter [...] Even the poorest tribes have produced work that gives them aesthetic pleasure [...] [They] devote much of their energy to the creation of works of beauty. (p. 9)

Jens Høyrup (2019) discusses the relationship between the geometrical structures (symmetries, in particular) that can be found in pottery, weaving, and other artefacts, and the development of formal geometry. With particular reference to the studies by Paulus Gerdes and his colleagues into the decorative art of Subsaharan Africa, he asserted that 'the decorations of many cultures [...] can be regarded in full right as expressions of formal investigation and experiment' (p. 202). Nevertheless, he cautioned that 'no necessity leads from an aesthetics of forms to formal investigation of forms' (p. 203). In any case, common to aesthetically motivated creations and formal mathematics is the idea of pattern (Mukhopadhyay, 2009).

The ludic impulse ('playing', in Bishop's list) likewise may be invoked as a wellspring of mathematical activity. In the earlier known recordings of mathematical activity, in such forms as cuneiform and papyri, are inscribed mathematical puzzles as well as data and practical problems. And besides puzzles, games of strategy and chance are also found across cultures. The attraction of intellectual play may be seen both in the popularity among general populations of puzzles such as crosswords (I can claim expertise in that field) and in the pursuit of 'pure' mathematics for its own sake. However, as Volker Runde (2003) reminds us, 'mathematicians live in the real world and their mathematics interacts with the real world in one way or another'. Which takes us to the next section...

State environments, socio-political constraints

> The functioning of the modern state presupposes a variety of mathematical technologies – accounting, statistics, and much more. Mathematics, on its part, needs the institutions of the state (schools, universities, research institutions, etc.) to secure financing, recruitment and the rearing of competence. (Høyrup, 2019, p. 635)

In a footnote, Høyrup further comments that, in the last four decades or so, 'without information technology, the immense increase of administrative control of citizens (to mention but that) would never have been possible'.

As societies became more complex, mathematics became a major resource for governance and statecraft. For example, Gary Urton (2009) discussed the complex mathematical resources that served administration of the Inkan Empire. In such examples, we see early examples of what Houman Harouni (2015) terms 'Commercial-Administrative Mathematics' (p. 59), dealing with finance, trade, censuses, labour, and citizenship.

Within Europe, as the Industrial Revolution gathered steam and thereafter, mathematics education was progressively tailored to produce a minimally trained workforce and to prepare people to live as practitioners or consumers of capitalism. Beyond Europe, it was 'the secret weapon of imperialism' (Bishop, 1990), and implicated in White supremacy, so cogently expressed in Høyrup's (2020) phrase 'the ideological shroud assigning the right to conquer and kill in the name of moral superiority' (p. 8).

At the beginning of the nineteenth century, Napoleon wrote that 'the advancement and perfection of mathematics are immediately connected with the prosperity of the state' (cited in Moritz, 1958). We find an echo in the *Executive Summary of the Final Report of the National Mathematics Advisory Panel* (2008), where it is stated that:

> During most of the 20th century, the United States possessed peerless mathematical prowess [...] But without substantial and sustained changes to its educational system, the United States will relinquish its leadership in the 21st century. Much of the commentary on mathematics and science in the United States focuses on national economic competitiveness and the economic well-being of citizens and enterprises. There is reason enough for concern about these matters, but it is yet more fundamental to recognize that the safety of the nation and the quality of life – not just the prosperity of the nation – are at issue. (p. xi)

This quotation exemplifies what President Eisenhower, in a draft of his retirement speech, referred to as the 'military-industrial-academic complex'. Mathematicians benefit from the perceived importance of their discipline, typically with scant acceptance or even awareness of moral responsibilities; enabling 'the unreasonable political effectiveness of "mathematics"'.

Another category proposed by Harouni, that of 'social-analytical mathematics' (p. 67) is exemplified in economics and social statistics.

A genealogical account of the development of mathematics of this kind, based on collecting vast amounts of data and creating conceptual and mathematical means for their analysis, inextricably intertwined with views on the nature of collective human behaviour within societies, was provided by Hacking (1990). Advances in information technology have immensely increased the ability to accumulate and process data and to build models that format many aspects of our lives (as pointed out by Ole Skovsmose for decades), models that are generally beyond the control of those affected and typically not even accessible to their inspection.

As for the relationship between mathematics education and governance, space does not permit even a minimal discussion, so I restrict myself to the following (adapted) aphorism: 'All education tends to control, and mathematics education tends to control absolutely.'

Internal/external drivers, and the two faces of mathematics

> Three main models have been traditionally used to explain scientific development and change. According to one, scientists respond to the results of earlier science and to questions raised by these results ('internalism'); according to another, general (mostly technological) social needs are the moving force, and their absence a brake (one brand of 'externalism'). The third approach [...] looks into the general history of ideas more specifically into the history of philosophy, for the causes that make scientists organise their search and shape their theories as they do. (Høyrup, 1994, p. 124)

Høyrup characterises the above as a simplistic, nevertheless convenient, scheme, and it is so applied in this chapter, simplified further by omitting explicit discussion of 'the third approach' though that does appear *passim* in relation to 'general history of ideas', in particular:

- the emergence of empirical science;
- Eurocentrism – more specifically, Grecocentrism;
- logical positivism and its extended family (discussed in many chapters of this book);
- structuralism (see below).

The two faces of mathematics mentioned in the introduction are reflected in the conventional opposition of 'pure' and 'applied'. In general, the external drivers bear more on applications, and the internal relate more to pure mathematics. Again, there are many interactions, such as the familiar observation that the 'purest' of mathematics turns out to have applications, often decades after its development – even for Hardy, for example (see Chapter 3, this volume). Hacking (2014, pp. 146–148) refers to the older term 'mixed mathematics' reflecting an area such as theoretical physics that is dependent on a combination of empirical investigations and mathematical modelling. Runde (2003) also offered an improvement on 'pure':

> Pure mathematics isn't pure: neither in the sense that it is removed from the real world, nor in the sense that its practitioners can ultimately avoid the moral questions faced by more applied scientists. It would much better be called 'theoretical mathematics'. (p. 3)

This also covers the point made by Hacking (2014, p. 9) that mathematics can be applied to (theoretical) mathematics also.

Modelling acts constitute the interface between the two faces of mathematics. The modelling cycle is often simplistically represented in terms of mathematisation of a situation, derivation of results within theoretical mathematics, interpretation back into the context, and a reality check possibly followed by revision of the model. To those elements should be added (at least) the motivations of the modellers, the adequacy of the assumptions on which the modes is based, the range of applicable mathematics to hand or derivable for the task at hand, communication of interpretations to interested groups.

Historically, modelling was first applied to physical phenomena, notably in cosmology and physics; more recently, particularly through harnessing the power of computer simulations, the modelling of social and political phenomena has become prevalent. For such phenomena, the assumptions on which the model is based become critical, are often extremely tenuous, and ideologically porous. Modelling physical and social phenomena may be broadly contrasted as manifesting 'unreasonable effectiveness' (Wigner, 1960) and 'reasonable ineffectiveness'.

Internal drivers: Creating

For internal drivers, another convenient distinction may be drawn between acts of creating, addressed in this section, and acts of organising, addressed in the next. As throughout the chapter, there is a concentration on strategically chosen aspects laying groundwork for arguments advanced in Chapter 13.

Extending the general notion of environments – physical, cultural, and political – appealed to in the section on external drivers, the first part here deals with the constructed environments within which people designated as 'mathematicians' carry on the activities that are recognised as 'doing mathematics'. Now primarily universities (historically also religious institutions, royal courts, intellectual salons, and other milieux), these also include settings outside the academy, notably military establishments, industry, and the corporate world.

Any study of the history of mathematics makes clear the importance of people running up against the puzzles created when their current ways of thinking cannot cope with what they are noticing. Galileo, for example, was intrigued that there are as many squared natural numbers as there are natural numbers, but it took nearly three centuries before Georg Cantor proposed a reconceptualisation that resolved the issue – and famously commented that 'I see it, but I don't believe it'.

The next focus is on mathematical creativity, highlighting certain intellectual aspects and mental processes, such as those described by George Pólya based on his observations of the behaviour of mathematicians, including himself. A particularly powerful weapon in the mathematician's armamentarium is a sensitive antenna for the perception of structure, in particular the same underlying structure in apparently different contexts. Formally such insights are termed isomorphisms, aphoristically by Henri Poincaré's characterisation of mathematics as the art of giving the same name to different things (Verhulst, 2012, p. 157).

Running through the whole story of human interplay between biological and cultural evolution is the role of material representations (Kaput & Schaffer, 2002). In particular, the impact of computers represents a fifth stage; for a seminal analysis, see Kaput, 1992.

Constructed environments, disciplinary norms

Over the last two millennia or more, with cultural variations, formal mathematics has emerged as a discipline. Analysis in any detail of how this happened in different cultures would require another book. Here I merely stipulate some 'boundary conditions' for such a work, beginning with a caution from Cullen (2009):

> Can we identify an activity in ancient China with a family resemblance to what would nowadays be called 'mathematics'? Or was there a self-conscious and publically recognized group of people in ancient China with a family resemblance to what would be called nowadays 'mathematicians'? (p. 593)

In a similar vein, Høyrup (2013) discussed the criteria that might be used to judge the appropriateness of the term 'Babylonian mathematicians' (concluding that there were some, even if a small minority). Fast-forwarding to the modern era, Karen Parshall (2009) traced the internationalisation of mathematics between 1800 and 1960.

Within the academy, the niches established/occupied by individual mathematicians are naturally diverse (the image comes to mind of Hardy at high table enjoying port and walnuts). In general, a mathematician with a university position has enough financial security to devote her/his time to research and teaching, and most of them do. Further, such an individual enjoys the support of a local and extended community – a very full discussion of such collective aspects will be found in Hersh and John-Steiner (2011). The specific case of the Bourbaki collective, an extreme example of a norm-dense subculture, is discussed below.

There is also the issue of how mathematics relates to other disciplines – most obviously physics, statistics, and computer science but also social sciences – through statistical and other forms of modelling (see Chapter 8, this volume). Further, what I term special-purpose constructed environments exist outside universities. Highly specific constructed environments that come to mind are the Manhattan Project to develop nuclear weapons, and the code-breaking team led by Alan Turing at Bletchley Park; current military applications include a great deal of Artificial Intelligence, for example to program drones so that they can, without human intervention, 'decide' to kill people.

The corporate business world also provides environments for mathematical work. Two examples that spring to mind are William Sealy Gossett, developer of the t-test while employed as the Head Experimental Brewer by Guinness, and Claude Shannon, who developed Information Theory while working for Bell Labs. In both cases, work initially driven by situated problems proved to be of much wider significance (as any psychology student knows).

Epistemological crises, conceptual change

Expanding mathematical knowledge is much more than mere accumulation; it is driven by conceptual restructuring. The history of mathematics is replete with examples of puzzlement. Epistemological crises may break in moments (relatively speaking) when the unthinkable becomes thinkable and the ineffable effable, but the ramifications can extend across centuries (for example, from Galileo to Cantor, referred to above), indeed millennia.

I begin by sketching the fascinatingly complex history of what people have meant by 'number'. Every (or at least, essentially every) culture makes use of counting, and does so in natural ways reflecting the affordances of the environment; beyond that complications ensue. Here I minimally comment on four epistemologically revolutionary extensions of what is meant by 'number', intimately tied to the basic arithmetical operations.

Natural numbers to positive rationals

It appears that for a long time, the conceptualisation of positive rationals remained tied to that of natural numbers. For the Greeks, for example, fractions intimately related to ratios and proportions, often in geometrical contexts. Cultural diversity is evident – why, for example, did the Egyptians and others restrict themselves almost entirely to unit fractions? The Mesopotamians developed procedures for division by fractions equivalent to the rule not infrequently taught to students today to 'invert and multiply', using table of reciprocals. And so on…

Positive numbers to directed numbers

I cannot resist beginning with the quotation:

> 3 – 8 is an impossibility, it requires you to take from 3 more than there is in 3, which is absurd.

The source of the above statement was neither someone writing centuries ago, nor a mathematical ignoramus. It was Augustus De Morgan (1806–1871), an eminent English mathematician, in his extremely interesting book called *Study and Difficulties of Mathematics* (De Morgan, 1831/1910).

While it is relatively easy to expand the domain of application of numbers to directed numbers for addition and subtraction, it took a very long time to agree on an explanation for something the poet W. H. Auden, in his *A Certain World* (1970), remembered from school:

> Minus times minus makes a plus
> The reason for this we need not discuss.

Rational numbers to real numbers

The realisation that, for example, the exact length of the diagonal of a unit square cannot be expressed as the ratio of two natural numbers required a reconceptualisation of number; the details are unclear in a historical record complicated by mythical stories. It is generally considered that a rigorous theory of irrational numbers was accomplished in the nineteenth century by Richard Dedekind, Cantor, and Karl Weierstrass.

Real numbers to complex numbers

The story of how complex numbers came to be accepted is even more fascinating. A key part was the invention of diagrams providing a representation for the numbers and arithmetical operations on them. And then there are quaternions, surreal numbers, on and on … *And they are all called 'numbers'*!

The above sketch primarily relates to the expansion of numbers within theoretical mathematics. Another perspective is that numbers are embedded within cultural matrices – in Urton's (1997) phrase, they have a 'social life'. In contrast to the formal structural analysis of numbers and the operations upon them (see Bourbaki discussion below), in human contexts multiplication and division are polysemous (Greer, 1992). Emphasis on what people *do with numbers*, whether for practical purposes, societal functioning, or for intellectual pleasure, contrasts with what I assert, without further elaboration, is the unproductive, arguably even meaningless, question 'Do numbers (of a specified type, especially negative, irrational, complex) *exist*?'.

Beyond arithmetic, parallel examples can easily be found illustrative of the points attempted in this chapter from the histories of other components of school mathematics: algebra, geometry, calculus, probability. Space allows only the briefest hints of how those discussions might go:

- For 2500 years, formal algebra ('rich in structure but weak in meaning' as René Thom put it) had little or no practical purpose (Høyrup, 2013). The familiar school algebra of today (satirised as 'the intensive study of the last three letters of the alphabet') is the product of a representationally driven development over millennia.
- As the familiar story goes, Euclid's *Elements* provided a model of the axiomatic method in mathematics until flaws were discovered and rectified by David Hilbert – at the cost of losing the simplicity of the original five axioms. And the problem of the fifth axiom, that bothered mathematicians (such as Omar Khayyam) for a very long time, finally was resolved (at least temporarily) by the emergence of non-Euclidean geometries. Further liberating reconceptualisations ensued, with the escape from a mere three dimensions to many, and on to Mandelbrot's exposition of fractal geometry. For some mathematicians (notably the Bourbakists), geometry became detached from its roots in locating and spatial cognition, and was absorbed into formalism.
- The story of calculus is long, and profoundly illustrates the importance of representations (Kaput, 1994). While its roots lie deep in intuitions of time and movement as continuous, it became a major topic for the nineteenth-century drive for

rigour. And it raises issues of intellectual priority between 'Europe' and India.

- Probability is a very special case, since its explicit mathematisation is relatively recent and hence more open to historical documentation and analysis. Hacking's (1990) remarkable work shows how its development was related to the most general social and political issues of statehood, mass collections of data, conceptions of the nature of humanity in the mass.

Across all of the branches of mathematics, there has occurred a revolutionary shift from the conception of mathematical formulations as providing, in some sense, a direct picture of the world, to the reconceptualisation that they model the world, in some sense.

There are many theoretical frameworks that may be invoked to explicate the above. In terms of theoretical mathematics, a pervasive need is for closure, in the technical sense. It is a prime driver in the expansion of 'number' to more and more complex structures. When addition and its inverse, subtraction, and multiplication and its inverse, division, arise through contemplation and applications of the natural numbers, the fact that subtraction and division are not always possible drives consideration of the possibility of negative and rational numbers and so on, for each expansion. At each stage, a local equilibrium is achieved (the real numbers, with a coherent representation in the number line, the complex numbers underpinning the fundamental theory of algebra) which itself harbours the germ of a disequilibrium. The parallel with a central aspect of Jean Piaget's account of cognitive development should be obvious.

Mathematical creativity

Throughout the history of mathematics, there have been individuals who have realised remarkable insights in posing and solving mathematical problems. The completion of a proof of Fermat's Last Theorem by Andrew Wiles and others arouses intense admiration among the general public, although, or perhaps because, the technical details are beyond all but a very small number of mathematicians; yet the pleasure of solving a problem insightfully is open to everyone, including children. On the assumption that the reader knows some or many of the canonical examples, the focus of this section is on how the individual's intellectual feats are embedded within the collective activities of communities of mathematicians.

To begin with a statement of the obvious, mathematicians approaching a creative challenge come forearmed with a great deal of resources – including methods, representations, proofs, structural analysis, problem-solving strategies, and so on (and they differ fundamentally from schoolchildren in these respects). These resources have been recorded, accumulated, critiqued, and systematised over centuries and multiple cultures. Mathematicians operate within the circles of their forebears and contemporaries, which, due to communicational advances, are now globally and speedily accessed.

These resources are activated by deploying a range of routine methods, heuristics, strategies, combined with mental flexibility. Above all is the disposition to look for and exploit structure. One of my teachers used to say 'Good mathematicians are lazy' by which he meant that they would look for an insightful rather than routine but laborious technique. (The apocryphal story of the young Gauss finding a 'smart' way to sum the integers from 1 to 100 is the classic example.)

Through observation and analysis of the behaviour of mathematicians, including himself, and a great variety of examples, Pólya inductively taxonomised some of these strategies. The following are among the most salient aspects of mathematicians' armamentaria:

- In its most explicit form, the exploitation of structure involves an isomorphism (Greer & Harel, 1998). A famous example is this account of a sudden insight:

> The idea came to me, apparently with nothing whatever in my previous thoughts having prepared me for it, that the transformations which I had used to define Fuchsian functions were identical with those of non-Euclidean geometry. (Poincaré, quoted in Newman, 1956, Vol. 4, p. 2020)

- By a kind of pattern recognition, before thinking about the details of a solution it is often possible to recognise problem/solution types, e.g., 'this kind of problem may well hinge on finding an invariant', 'clearly this can be handled by induction', and so on.

- Intuition is very frequently invoked by mathematicians. An agreed-upon definition, let alone a convincing theoretical explanation remains elusive (Fischbein, 1987). For present purposes, I take it to mean 'any immediate inference in which there is no conscious reasoning' (Hacking, 2014, p. 17).

Beyond the deployment of these resources, in ways that may be more or less routinised, there are the most elevated forms of creativity when an individual or group achieves a conceptual restructuring, finds a hitherto unknown proof that goes to the structural heart of a big idea – or designs a transformative representation.

Material representations

Material representations have been of crucial importance in the creation, accumulation, organisation, and communication of mathematical results. As throughout this chapter, an attempt is made to use space efficiently through powerful examples. In this section, the focus is on inscriptions on paper and other materials (in particular notations), and diagrams. A separate section, which follows, outlines the revolutionarily new resources afforded by advances in computer-based representations (Kaput, 1992). No attempt is made to address the vast topic of natural language and mathematics or that of mental representations.

A human starting point is the body; it is no accident that the most common bases for numerical systems are 5, 10, 20; some cultures go beyond manual and pedal digits. Body parts are also ubiquitous in measurement (hand, foot, cubit...). And in recent years much attention has been given to embodied cognition.

If it is helpful – which I doubt – to speak of mathematics as a language, then it is one that draws on natural languages, with enhancements, and with particular notations. A glance at the encyclopaedic work of Florian Cajori (e.g., 1928–1929/1993) is enough to make clear how rich and complex, messy and arbitrary, has been the evolution of such. A familiar example of how instrumental a good notational representation can be is the contrast between the user-friendliness of decimal numbers for purposes of calculation and the system used by the Romans. For a more advanced example, De Morgan (1910, p. 185), citing Pierre-Simon Laplace and referring to notation for powers, wrote:

> Newton extended to fractional and negative powers the analytical expression which he had found for whole and positive ones. You see in their extension one of the great advantages of algebraic language which expresses truths much more general than those which were at first contemplated [...]

(which stands in marked contrast to his blinkered view on 3 – 8 cited above). Thus, the power of the notation x^n is that it opens up the possibility of conceiving of other values of n, eventually leading to the remarkable equation $e^{i\pi} = -1$ (Lakoff & Núñez, 2000, p. 433).

Graphical representations, naturally enough, are central to geometry, combined with the conventions for using letters to label elements. Here may be mentioned the distinctive position of Reviel Netz, emphasising what Bruno Latour (2008, p. 3) called 'scripto-visual inventions':

> I will argue that the two main tools for the shaping of deduction were the diagram, on the one hand, and the mathematical language on the other hand. Diagrams – in the specific way they are used in Greek mathematics – are the Greek mathematical way of tapping human visual cognitive resources. Greek mathematical language is a way of tapping human linguistic resources [...] But note that there is nothing universal about the precise shape of such cognitive methods. They are not neural; they are a historical construct. [...] One needs studies in cognitive history, and I offer here one such study. (Netz, 2003, pp. 6–7)

The fusion of geometry and algebra was, of course, a revolutionary passage in the history of mathematics, heavily dependent on the invention of Cartesian graphs. In similar vein, an extended analysis of the long history of representations in the development of calculus was provided by James Kaput (1994). And, as Kaput (1992) has pointed out, a fundamental level-shift in material representations lies between those which record and those which are manipulable, for example for executing calculations (e.g., the abacus or the Quechuan *yupana*). Computers have taken representational resources to new levels, as outlined next.

Computers: Opening new representational windows

Computers and associated technologies have significantly changed the doing of, and the conception of, mathematics in multiple ways, including the following:

- Most obviously, increase in brute computational power as exploited, for example, in testing conjectures and, generally, leading to the acceptance that an empirical element may enter mathematics.
- The theoretical notion of computability, captured in the conceptual device of the Turing machine, leading inexorably to analyses of the limitations of computability.
- Changes in the conception of proof prompted by computer proofs and discussions – philosophical and practical – about their status. Hacking (2014) discusses the debate over whether a totally computerised proof machine will ever be possible.

Perhaps most importantly, computers have provided more powerful representational resources. Benoît Mandelbrot, the originator of fractal geometry (a creative feat hard to imagine possible before the computer era), commented that 'computers have put the eye back into computing'. A small sample:

- Being able to represent continuous change in a perceptually direct way, thereby moving 'past the algebra bottleneck' (Kaput, 1998, p. 278) *en route* to calculus.
- The representation of geometrical procedures, not just diagrams. For example, consider the theorem that joining the midpoints of the sides of any quadrilateral produces a parallelogram. Using Geometer's Sketchpad, the user can store a procedure (not a static image) corresponding to that result. Then, any vertex of the quadrilateral can be 'grabbed' by the cursor and moved, and the whole configuration moves accordingly; it should be clear that this gives a whole new insight into the invariance at the centre of the theorem.
- Generativity, as shown *par exemple* in the simplicity of the Turing machine, relative to the huge mathematical edifice that can be built on that foundation. Another example is the Logo programming language built on the two primitives of moving forward a certain distance, and rotating through a certain angle. The language affords construction of a hierarchy of procedures building on procedures.

- Computers allow the display of much more complex data, qualitatively as well as quantitatively for example data varying over time, and for realistic modelling, for example the software STELLA, which enables school students to build, run, and evaluate system dynamical models (e.g., Fisher, 2021).

Internal drivers: Systematising

> A great part of mathematical activity today is organizing [...] When compared with creating, organizing scientific cognition seems to be an inferior activity. Yet [...] in no science are these two activities so densely interwoven as they are in mathematics. (Freudenthal, 1973, p. 414)

The sheer volume of established mathematical knowledge now is such that consolidating it as a coherent body of knowledge and techniques is a daunting task, even if restricted to 'pure' mathematics, as is the focus of this section. Material resources deployed in the attempt include inscriptions, notational systems, taxonomies, books that survey the field, classical textbooks. Internally, there are definitions, axioms, theorems, visual representations, structures...

Does it make sense to speak, as Nicolas Bourbaki (1950) did, of 'the architecture of mathematics'? Arguments against that are advanced below. Admittedly, there is considerable agreement on what is accredited within theoretical mathematics (using that term instead of 'pure'). Rejecting teleology as I do, thereby refusing to accept a forced choice between the development of mathematics being inevitable or contingent, the position of Rafael Núñez (2000) seems appropriate:

> Mathematics is not transcendentally objective, but it is not arbitrary either (not the result of pure social conventions). (p. 3)

In support of this position, examples are cited where some aspects of codified mathematics seem inevitable, being tied to the human condition, and reflecting a hard-to-deny internal coherence. Other aspects are contingent, reflecting environmental and cultural diversity, the impact of external events, technological developments, specific individual and collective creative acts. Relevant also is the evolutionary perspective; mathematics generated is subject to selection processes, only the fittest surviving.

Three aspects central to the systematisation of theoretical mathematics are abstraction from its human roots, rigour, and structure. The drive for rigour is perhaps the most defining characteristic of European mathematics in the nineteenth and twentieth centuries. And the complex notion of structure (with loose and disputed ties to the amorphous movement called 'structuralism'), does provide a systematic summarising of a great deal of theoretical mathematics, which in turn constitutes a very powerful resource for mathematicians advancing the field.

All of those aspects are clearly exemplified in the Bourbaki movement of the twentieth century which made a heroic, but arguably doomed, attempt to define the architecture of which Jean Dieudonné spoke.

Historico-genetic development of mathematics: Inevitable and contingent

> A repeating process, an interplay of form and content, which characterizes mathematical thought (Freudenthal, 1991, p. 10)

Explaining why he adopted the term 'anthropology of mathematics' to characterise his scholarly field, Høyrup (1994) stated:

> What I looked for was a term which suggested neither crushing of the socially and historically particular nor the oblivion of the search for possible more general structures: a term which neither implied that the history of mathematics was nothing but the gradual but unilinear discovery of ever-existing Platonic truths nor [...] a random walk [among] an infinity of possible systems of belief. A term, finally, which involved the importance of cross-cultural comparisons. (p. xi)

(And see the quotation from Núñez above.)

The question 'Is the development of mathematics inevitable or contingent?' presents, in my view, a false choice. There are, indeed, aspects of the development of mathematics that it is hard to imagine happening otherwise. Arguably the clearest example is 'number' as the usages of the word developed over many centuries, from the naturally termed 'natural numbers'. As Freudenthal (1991) put it:

> The first non-trivial structure as such, i.e. whole number as the product of the process of counting, begot rich process and product content which,

> organised by ever new structures, in turn begot new contents – a never ending cyclic process. (p. 10)

Mathematics has to be generative for the same reasons that language is generative. As more complex societies evolved, the practical need for dealing with large numbers meant that structural intervention became necessary in order to avoid the fate of Jorge Luis Borges' character 'Funes the Memorious', who had a separate image and name for every natural number. The specifics may be contingent, but the emergence of *some* such construction seems inevitable.

Further, there are many aspects of mathematics that make it difficult to disagree with Freudenthal (1991) when he stated that 'mathematics grows, as it were, by a self-organizing momentum' (p. 15). Again, think of numbers, and the simple example of going beyond whole numbers to fractions, motivated by so many practical situations. Of course, this took many centuries, with great cultural variation. The extension to directed (negative as well as positive) took even longer to bring to the point of formal respectability, although people managed much earlier to deal with practicalities such as debt.

In a very thorough and nuanced discussion of the issue, Hacking (2014) declares his support for what he calls 'the Latin model', the name being suggested by an analogy of the evolution of Latin into Romance languages – contingent in detail, but subject to significant constraints. He also argued that, while 'our notion of the infinite was not inevitable [...], our notion of complex numbers was inevitable' (pp. 117–121). It is hard to see the development of numbers beyond the counting numbers as other than inevitable, but that is clear only insofar as *some* such development was driven by practical, and also supra-utilitarian, needs. But inevitable to what degree? Fractions and negative numbers, surely, but complex numbers? Quaternions and octonions? Conway's surreal numbers? And when studying the natural numbers as a system became an interest, was it inevitable that prime numbers should take such a central role? Perhaps, but how about other named numbers with special properties given poetically suggestive names – 'perfect', 'amicable', and so on?

The formalisation and systematisation of mathematics accomplishes a great deal in terms of generativeness. From the five axioms of Euclid, a huge edifice can be constructed (albeit Hilbert pointed out cracks a long time later). The definition of a group is simple (see next section) yet on that foundation, again, so much can be constructed, including the recently completed classification of finite groups.

Have you noticed that when people want to argue that mathematics is universal and certain, they use simple examples, such as 'the angles of a triangle add up to 180°' or $2 + 2 = 4$? With appropriate clarification, it's hard to argue with either statement. But similar statements about, for example, non-standard analysis are not at all clear. And there is no such simplicity or obviousness about probability, in particular *subjective* probability (Devlin, 2014). It is a norm within academic mathematics to take proof as central, yet Hacking (2014) was prepared to argue that 'deep mathematics could have developed without proof at all' (p. 115).

'Self-organising' may be interpreted in terms of each local equilibrium containing within itself the germ of disequilibrium. Studying the real numbers, mathematicians, from at least the Babylonians, became interested in quadratic and cubic equations. Throw in the apparently very strong psychological need for closure, in the mathematical sense, and eventually the need for positing the square root of –1 became tempting though frightening, then it appeared to work, eventually it became formally ratified.

Thus, time-dependency must be acknowledged. What appears inevitable in hindsight was certainly not so during the struggles for epistemological coherence. And the temptation to believe in teleology, implying the possibility of a definitive characterisation of mathematics, does not hold up. In his critique of the Bourbaki-Piaget axis (see Chapter 13), Freudenthal (1973) stated as follows:

> Piaget is not a mathematician, so he could not know how unreliable mathematical system builders are [...] Mathematics is never finished – anyone who worships a certain system of mathematics should take heed of this advice. (p. 46)

Or, as Høyrup (1995) put it:

> No critique is ever definitive. What seemed at one moment to be an absolute underpinning [...] turns out with historical insight to make other 'naïve' presuppositions which in their turn can be 'criticized'. (p. 5)

The discipline of the discipline: Abstraction, rigour, proof, structures

Dominant themes of (European and extended-European) theoretical mathematics in the nineteenth and twentieth centuries were: increased abstraction; the drive for rigour, yoked to the desire to establish unassailable proofs; specification of a structural architecture – all manifestations of a craving for absolute certainty, as was the dream of establishing mathematics on logic, discussed in various chapters of this volume. Elements of this quest include the chimerical search for absolute definitional precision, the power of axioms, impeccably formal proofs, and a network of abstract structures, replacing the problematic metaphor 'mathematical objects' with webs of relations among undefined entities. Discussion of abstraction and rigour will be found in various chapters of this book (and see the next section, on Bourbaki); some key points about proof and on general and specific notions of structure, follow.

A great deal of Hacking (2014) is concerned with proof. In particular, he makes a clear distinction between 'two visions of proof' (p. 11) which he labels with the names of two mathematical greats:

> There are proofs that, after some reflection and study, one totally understands, and can get in one's mind 'all at once'. That's Descartes.
>
> There are proofs in which every step is meticulously laid out, and can be checked, line by line, in a mechanical way. That's Leibniz.

Leibnizian proof is the dominant image of how people do proofs, reinforced by the norm of publishing mathematical papers whereby all traces of how the proof was found are expunged.

One characterisation of mathematics is as 'the study of all possible patterns'; patterns may be thought of as partial manifestations of the rigorously defined structures of modern mathematics. The pattern of addition and subtraction of even and odd integers (even + even = even, etc.) is accessible to quite young children; formally, this pattern is a feature of one instantiation of a group with two elements. The concept of a group is simple to define, yet with immense ramifications both in terms of modelling situations and in terms of the architecture of formal mathematics. A group is defined as the coupling of a set, S (which may have a finite or infinite number of elements), and an operation, S, applicable to any two elements of S and having certain properties (the list is redundant relative to minimal definitional requirements), including the following:

- Closure: For any two elements of S, x and y, $x \circ y$ exists and is an element of S.
- Identity element: S contains a unique element, e, such that for any x, $x \circ e = x$ and $e \circ x = x$.
- Inverse: for any x, there is another element x^{-1} in S such that $x \circ x^{-1} = x^{-1} \circ x = e$.

All of these properties relate to extremely pervasive aspects of mathematics.

The history of how this axiomatisation crystallised out of multiple, apparently unrelated, situations that could be modelled by groups is a fascinating episode in the history of mathematics. In arithmetic, the (positive and negative) integers, rationals, real numbers, and complex numbers, with the operation of addition, form groups, for example. In geometry, systems of transformations form groups. Galois theory in algebra is based on group theory. Groups are central to the theory of crystallography. They have been invoked by Piaget and Claude Lévi-Strauss, and are pervasive in the work of M. C. Escher. Rubik's cube was designed to help teach group theory. Groups and other structures such as rings and fields are central to the proposed architecture of mathematics as envisaged by the Bourbaki collective, to which we next turn.

The case of Bourbaki

> The most spectacular example of organizing mathematics is, of course, Bourbaki. (Freudenthal, 1973, p. 46)

There are two central reasons for including this section. First, as expressed in the quotation above, the Bourbaki project stands as the supreme attempt to deliver an organisation for *selected* parts of mathematics (excluding applied mathematics, probability theory, and much else). Second, as is taken up in Chapter 13, the influence of Bourbaki (not always emanating from Bourbaki itself) spread into mathematics education, with continuing and arguably harmful ramifications.

Bourbaki also serves as probably the most extreme example of a constructed environment within which self-described 'working mathematicians' could do mathematics, one whose origins can be traced back to the aftermath of the First World War, when the ranks of French academic mathematicians were depleted due to many of them falling in the war, and survivors wished to restore the standing of *French* mathematics. The very distinctive organisation of Bourbaki as a kind of secret society (Mashaal, 2006), or club, is well summarised in the Wikipedia entry (and see Hersh & John-Steiner, 2011, pp. 181–191). What is clear is that their effort to systematise mathematics in an uncompromisingly formalist style based on defining mathematical structures was extremely influential on the field through much of the twentieth century, with residual influence to this day. The following, from one of the most distinguished mathematicians of the recent past, strikes me as a balanced view:

> All mathematicians of my generation, and even those of subsequent decades, were aware of Nicolas Bourbaki, the Napoleonic general whose reincarnation as a radical group of young French mathematicians was to make such a mark on the mathematical world. His memory may now have faded, the books are old and yellowed, but his influence lives on. Many of us were enthusiastic disciples of Bourbaki, believing that he had reinvigorated the mathematics of the twentieth century and given it direction. But others believed that Bourbaki's influence had been pernicious and narrow, confining mathematics behind walls of rigour, and cutting off its external sources of inspiration. (Atiyah, 2007, p. 1150)

Atiyah neatly underlines the last point by pointing out that 'had Euler worried too much about rigour, mathematics would have suffered' (p. 1151).

While most emphasis is on the collective aspect of the Bourbaki mathematicians, Gerhard Heinzmann and Jean Petitot (2020) clarify that Bourbaki 'was at the same time the collective author of a monumental and long-lasting treatise [...] and a pleiad of individual geniuses [...] who were at the cutting edge of innovation and creativity' (pp. 187–188). Heinzmann and Petitot also emphasise the view within the collective persona of Bourbaki that they were providing a powerful toolbox to facilitate the creativity of 'working mathematicians'. They also point to a central Bourbakian tenet of the unity of mathematics, as implied by the singular form in the title of their treatise *Elements de Mathématique*, and as manifest in so many examples of structural connexions across diverse branches of mathematics. Hacking (2014, p. 13) stated that 'the history of mathematics is one of diversification and unity', so that when, for example, Descartes brought together geometry and algebra, they turned out to be 'the same stuff' (p. 11).

Other mathematicians diverged from Bourbaki, including Thom (1971) and Mandelbrot (2002). Alexander Grothendieck proposed a new organisation around category theory that was not taken on board. Mandelbrot (2002) convincingly argues that Bourbaki's history was shaped by a series of historical accidents that they never acknowledged, believing themselves to be 'the necessary and inevitable response to the call of history' (p. 31).

The attenuated but continuing impact of Bourbaki on school mathematics is discussed in Chapter 13. Here, for the sake of brevity, I point to some facets of what I see as the supreme irony of Bourbaki – the contrast between its adherence to *mathematical* rigour and the irrationality and contradictions of its philosophical and socio-political stances:

- *Universal versus chauvinistic mathematics*: The image of mathematics venerated within Bourbaki is universal, yet the organisation of Bourbaki as a constructed environment within which to systematise mathematics was decidedly French in terms of original motivation, membership (predominantly), and style.
- *Cavalier attitude to philosophy*: As expressed by Reuben Hersh most 'working mathematicians' do not fret over philosophical issues. Dieudonné (1970) made a similar comment:

> On foundations we believe in the reality of mathematics, but of course when philosophers attack us with their paradoxes we rush to hide behind formalism [...]. Finally, we are left in peace to go back to our mathematics and do it as we have always done, with the feeling each mathematician has that he is working with something real.

- Mandelbrot (2002) stated that Bourbaki had 'only contempt for the logical foundations of mathematics', such as the work of Kurt Gödel and Turing (p. 31).

- *Bizarre claim to include all of mathematics*: in their series of textbooks (more accurately described as an encyclopaedia, as Leo Corry (2009) has pointed out) they claimed to be surveying the whole of modern mathematics, despite totally excluding applied mathematics, any connection with physics, and also probability.

On a more specific point, their taboo against diagrams (while allowing themselves poetic licence in choosing technical vocabulary) is hard to understand, and I have found no clear explanation for that. It is hardly surprising, then, that mathematicians such as Thom and Mandelbrot were alienated.

Their claimed liaisons with other manifestations of the general cultural movement, structuralism (or, to be careful, other uses of the term) seem opportunistic – in particular, their rather one-sided romance with Piaget. Hacking (2014) proposed a clear distinction between 'mathematician's structuralism' and the structuralism of recent analytic philosophy (p. 237). Kantor (2011) unequivocally characterised the supposed relationship of Bourbaki's structures to structuralism as 'pure intellectual fraud' and he elaborated that 'referring to Bourbaki in structuralist essays was a way of giving some scientific credit and weight to works of variable quality' (and see Aubin, 1997).

The Bourbakists were, of course, entitled to define 'mathematics' as they wished, essentially ignoring one of its faces, as long as the definition was clear, which it was. However, it is arguable that they influenced the image of mathematics among mathematicians and non-mathematicians in an unbalanced way, which had harmful effects on mathematics education during the New Math period and continuing to this day.

Looking back and forward

Central to this book is disruption of the tendency to take for granted mathematics-as-discipline, mathematics-as-school-subject, and the relations between them; to that end, this chapter is intended to support arguments advanced in the intimately related Chapter 13. A necessarily broad-brush sketch of the history of people developing mathematics has been attempted, with a simplifying framework of distinguishing between external and internal drivers, and between acts of creating and acts of systematising.

The history of mathematics continues to happen. Looking into the future, a clearer picture of the powers and limitations of computers and Artificial Intelligence will emerge. There is no lack of unfinished business from the past. When Hilbert, in 1900, set out twenty-three mathematical problems to be solved in the twentieth century, the continuum hypothesis was the first. This unproved hypothesis relates to the cardinality of the real numbers, which, since Cantor, is known to be greater than that of the natural numbers (or the rationals) but it remains unknown whether there are any intermediate cardinalities. A major theorem which may help to settle the issue was recently published (Asperó & Schindler, 2021). This example serves admirably to show that many questions within mathematics remain open, as does the Wikipedia summary of the current consensus among mathematicians in relation to the Hilbert's problems as to whether they have been solved, remain unsolved, or were not stated with sufficient precision.

Among the overarching themes in this chapter selected for the framing of Chapter 13 are: the conception of environments from physical through socially constructed, the emergence of mathematics and mathematicians as identifiable collective activities and actors, the two faces of mathematics, the centrality of epistemological shocks and their resolutions, the defining characteristic of mathematicians to look for and exploit patterns, and the role of material representations.

Key issues to be addressed in Chapter 13, with references back to this chapter, include:

- The relevance of history of mathematics to school mathematics, rejecting any simplistic interpretation of 'ontogeny recapitulates phylogeny'. In particular, what can be learned from epistemological crises and their resolutions to figure out how to help children through the radical reconceptualisations they need to negotiate.
- The embeddedness of mathematics in culture, despite historical disembeddings, which has massive implications for school mathematics.

- Absolutely central point: the child learns *under instruction* – in a constructed environment, insofar as school learning is concerned. This is, perhaps, the fault-line between Freudenthal and Piaget. I will argue that the latter's idea of some sort of 'natural development', and variations on that theme by radical constructivists, do not bear examination. Piaget was impressed by someone figuring out that the cardinality of a set of objects is independent of the order of counting but offered no account, as far as I can see, of how to get from there to, say, the solution of a quadratic equation.
- In particular, the supposed correspondence between the mother-structures of Bourbaki and the structures of Piaget's developmental theory, and the educational damage that resulted, will be addressed. The waning direct influence of Bourbaki does not mean that it is dead. A contemporary curricular framework (for which I suggest the Common Core State Standards in the US affords a representative example) could be characterised as 'Bourbaki light' – a sequence structured in a superficially 'logical' form.
- School mathematics is predominantly presented as pre-systematised, with little opportunity for students to experience systematising, let alone creating.
- In relation to mathematical modelling, it will be argued that school mathematics, in general, fails to deal with the core issues. A long-term research program on word problems feeds directly into this discussion, and a thread can be followed from there to all the work on formatting and so on.
- Discussion of Bourbaki naturally raises the question of selection. Out of all the mathematics now assembled and organised, what should be selected for children to learn in school? Some possibly iconoclastic ideas for curriculum and pedagogy will be presented.
- It will be argued that academic mathematicians enjoy too much power to influence how school mathematics is framed and done.

- Elementary mathematics education is foundational, not just for later mathematics education, but in the framing of an individual's worldview; it will be argued that school mathematics, as typically practiced, tends to produce a destructive image of mathematics.

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