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BREAKING IMAGES

ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

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3. Hardy's deep sigh

Ole Skovsmose

In his book A Mathematician's Apology, *Godfrey H. Hardy presents a conception of mathematics according to which real mathematics can be considered harmless and innocent. By 'real' mathematics, Hardy has in mind, for instance, advanced number theory. He contrasts real mathematics with different examples of applied mathematics and cases of elementary mathematics. Hardy argues for the thesis of innocence by asserting that the utilitarian value of real mathematics is nil. Real mathematics does not have any useful applications. By assuming a utilitarian perspective on ethics, Hardy can claim that real mathematics operates at a comfortable distance from any ethnical and political controversies. However, number theory, that Hardy considered the epitome of real mathematics, has tremendous applications itself within war technology. Hardy's explicit justification of the thesis of innocence is simply fallacious. Most ironically, the doctrine of neutrality continues to operate. According to this doctrine, mathematics can be researched and developed while ignoring any kind of ethical and socio-political considerations. The doctrine of neutrality becomes acted out through mathematical research paradigms, dominating the vast majority of university departments in mathematics the world over.*

By the turn of the nineteenth century, science and technology were seen as motors of progress. As part of the Western outlook, it was broadly assumed that science and technology ensure welfare in all aspects of life, whether we are dealing with material production, economic resources, health care, or education. The organisation in 1851 of the Great Exhibition in the Crystal Palace in London, the erection in 1899 of the Eiffel Tower, and the presentation in 1900 of the Exposition Universelle in Paris symbolise the optimism that dominated the whole era. The very steel material used for the construction of the Eiffel Tower and the steel

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and glass used for the Crystal Palace announce the potentials of the coming century.

Naturally, this optimistic celebration of progress presupposed that a range of socio-political and economic factors were ignored. The horrible living conditions of the working class in industrialised countries came to be seen as unavoidable and, therefore, ignorable necessities for the modern world order. The broadly assumed racist outlook ensured that the brutality of colonialism was ignored as well. By the turn of the century, people, in particular those belonging to the well-protected layers of Western societies, could enjoy reading about world exhibitions—if not in fact going there—and be contented by living during a period of assumed ongoing progress.

Such visions of the future were shattered by the outbreak of the First World War. This catastrophe revealed a new dramatic connection between, on the one hand, science and technology, and on the other hand, war. While science and technology were supposed to constitute an integral part of peaceful and enlightened progress, they now appeared also as an integral part of the very machinery of war. The development of new and more powerful weapons was a science-based technological achievement. Submarines and airplanes became indispensable components of warfare. The application of poison gas likewise brought chemistry to the forefront of the battlefield. The First World War made evident that the image of science and technology as reliable motors of peaceful progress was an illusion.

A life

In 1940, as the Second World War was in dramatic development, Godfrey H. Hardy published the book *A Mathematician's Apology*. Rather than reading it as an immediate reaction to the outbreak of that war it could be seen as a profound, but delayed, reaction to the First World War. In the inaugural lecture that Hardy gave in Oxford in 1920, one finds an 'outline of an apology for mathematics' (Hardy, 1967, p. 74); so Hardy's first 'apology' was formulated long before *A Mathematician's Apology* was published. The First World War put the relationship between mathematics and war on the agenda, and certainly also on Hardy's agenda.

Hardy was born in 1877. In school, he was not particularly dedicated to mathematics, but from his early years demonstrated excellence in the subject. Hardy related that he primarily thought of mathematics in terms of competition, and found that there he could most decisively beat others.

In 1896, he entered Trinity College in Cambridge to study mathematics, and in 1900, he became a fellow. In 1898, he became a member of the Apostles, which was a closed elitist discussion group that also included George Moore (1873–1958), John Maynard Keynes (1883–1946), and Bertrand Russell (1872–1970). The Apostles was open only to brilliant scholars from the University of Cambridge, and at their meetings any topic could be addressed. The most famous non-member of the Apostles was Ludwig Wittgenstein (1889–1951), who was invited to join but did not find the group serious enough. Like Keynes and Russell, Hardy also joined the Bloomsbury Group, which focused on literature and art. Hardy was well located in the academic and intellectual circles at the time, and was aware of the current and controversial issues being discussed, in relation to politics, literature, or art. In 1906, he secured a position as lecturer in mathematics in Cambridge, and during the First World War he preoccupied himself with his teaching and research.

Russell was a declared pacifist, revolted by the English jingoism that accompanied the outbreak of the First World War. Hardy was not outspoken with respect to political issues, but well aware of Russell's sentiments. Russell held a position as lecturer at Trinity College, but in 1916 he was dismissed from this position as a consequence of his anti-war writings. In 1918, he was put in prison for five months, and during that time he wrote *An Introduction to the Philosophy of Mathematics* (Russell, 1919/1993). Hardy shared Russell's anti-war positions, and during the war he felt more and more uncomfortable staying in Cambridge, where jingoism was strongly articulated by some of his colleagues.

In 1919, Hardy took up a professorship in Oxford, and was received with enthusiasm by the younger mathematicians there. That he felt it important in his inaugural lecture to outline a defence of mathematics can come as no surprise. The atrocities of the First World War, and the roles played by mathematics, made such an apologia necessary. Its presentation made it possible for Hardy to concentrate completely on mathematical research, and the next ten years were very productive for him. In particular, his work with John Littlewood and Srinivasa Ramanujan became one of the outstanding collaborations in the history of mathematics.

A photo of Vladimir Lenin was displayed on the wall of Hardy's room in New College, Oxford. This information is noted by C. P. Snow, who wrote a biographical sketch of Hardy as preface to *A Mathematician's Apology*. I am not aware of any explanation of Hardy's choice of photo, but one should not conclude that Hardy was a communist. If he had leftist inclinations, they likely reflected a non-standard interpretation of the term. At that time, many intellectuals in England demonstrated an open curiosity for what was taking place in the Soviet Union.

In 1920, Russell visited the Soviet Union as a member of a British delegation and, during the visit, had the opportunity to meet Lenin in person. Russell became disillusioned, and back home he wrote a critique of what he saw: *The Practice and Theory of Bolshevism* (Russell, 1920/2017). However, Russell's critique was based on a profound political sympathy, and he states: 'The existing capitalist system is doomed. Its injustice is so glaring that only ignorance and tradition could lead wage-earners to tolerate it' (p. 2). I assume that Hardy had read Russell's book, and the picture of Lenin might represent some feeling of resonance.

Hardy was certainly in full accord with Russell's attacks on Christianity. In 1927, Russell gave the lecture 'Why I Am Not a Christian', wherein, among other things, he states that 'every single bit of progress in human feeling, every improvement in the criminal law, every step towards the diminution of war, every step toward better treatment of the coloured race, or every mitigation of slavery, every moral progress that there has been in the world, has been consistently opposed by the organized churches of the world' (pp. 20–21). The lecture was circulated as a pamphlet, and later included in several books as, for instance, Russell (1957). Hardy shared Russell's anti-Christian stance. He did not go to church, quite literally: he simply did not enter a church under any circumstances, not even when requested to do so for academic ceremonies.

When Hardy felt that his creative mathematical powers had declines, he experienced periods of post-creative depression. These moments provided the personal context that ultimately led him to write *A Mathematician's Apology*. It opens with the following statement:

'It is a melancholy experience for a professional mathematician to find himself writing about mathematics' (p. 61). Hardy considered this type of writing second-rate work. He thought of writing about literature, theatre, as inferior activities: 'Exposition, criticism, appreciation, is work for second-rate minds' (p. 61). Presumably, Hardy had postponed this activity until he had no better things to do.

In 1941, Hardy published the booklet *Bertrand Russell and Trinity* (Hardy, 1970), in which he provides an account of Russell's dismissal from Trinity College in 1916. In the booklet, Hardy also gives glimpses of his own position, and he mentions that he had been secretary of the Cambridge branch of the Union of Democratic Control, founded shortly after the outbreak of the First World War. This was an organisation that represented war-sceptic positions. Hardy's insider clarification of what took place in 1916 only appeared twenty-five years after the event. I have no doubt that Hardy maintained clear priorities in life: first things first, and mathematics was a clear number one. Only after his creative powers had left, and he had written his apology, did he find time for clarifying what he felt to be a grave injustice done to his friend Bertrand Russell. After publishing *Bertrand Russell and Trinity*, Hardy published nothing more. He died in 1947.

A mathematician

In mathematics, Hardy worked in close collaboration with others. During most of his career, he collaborated with John Littlewood (1887–1977), who had entered Trinity College in 1903. Together they published more than 100 papers. Hardy also established a collaboration with the Indian mathematician Srinivasa Ramanujan (1887–1920), and together they published several papers.

Much of Hardy and Littlewood's collaboration was in number theory, for instance about the distribution of prime numbers. It appears common sense to consider their density to be decreasing in the sense that one could expect the number of primes between, say, 18000 and 19000 to be smaller than the number of primes between 8000 and 9000. Since Antiquity, it has been known that the number of primes is infinite, so their decreasing density will never reach zero. The prime number theorem provided an estimation of how the density decreases, and this estimation was first proposed by Carl Friedrich Gauss (1777–1855).

One can also consider prime twins, pairs of primes like 11 and 13, 41 and 43, and 107 and 109 that differ by 2. However, as the density of primes is decreasing, one could expect that the space between primes will be ever-increasing with the possibility that there is a largest pair of prime twins. However, according to the prime twin conjecture, there exist infinitely many prime twins, with decreasing density. Hardy and Littlewood provided an estimation of how this density decreases, similar in nature to the one provided by Gauss for prime numbers.

With respect to his start as a mathematician at Trinity, which one can link to around the year 1900 when he became a fellow, Hardy (1967) states: 'I wrote a great deal during the next ten years, but very little of any importance; there are not more than four of five papers which I can still remember with some satisfaction' (p. 147). The important turns in Hardy's career came in 1911 when he started his collaboration with Littlewood, and in 1913 when he came to know Ramanujan. He wrote that 'All my best work since then has been bound up with theirs, and it is obvious that my association with them was the decisive event of my life' (p. 148). Then follows an emotional remark: 'I still say to myself when I am depressed, and find myself forced to listen to pompous and tiresome people, "Well, I have done one thing that *you* could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms"' (p. 148, italics in original).

I like very much his addition 'on something like equal terms'. Hardy fully recognises that Littlewood and Ramanujan, both ten years younger than him, are mathematical geniuses. He is certainly also aware of his own unique creative powers. With both honesty and satisfaction, he can claim that he has co-operated with them – not at equal terms – but on something like that.

An apology

In *A Mathematician's Apology*, Hardy (1967) presents a conception of mathematics which we can think of as Hardy's working philosophy of mathematics.[1] Throughout all his formulations, he expresses a clear Platonic outlook:

1 See Chapter 1 in this volume for an introduction to the notion of 'working philosophy of mathematics'.

> I believe that mathematical reality lies outside us, that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply notes of our observations. This view has been held, in one form or another, by many philosophers of high reputation from Plato onwards, and I shall use the language which is natural to a man who holds it. (pp. 123–124).

In his research, Hardy sees himself as making discoveries, as for instance with respect to the distribution of prime twins. Many research mathematicians operate with Platonism as an implicitly assumed element of their conception of mathematics. This observation has been elegantly captured by Reuben Hersh (1997), when he states that mathematicians are formalist on weekends while Platonist during working hours. Hardy, however, was very aware of actual trends and positions in the philosophy of mathematics. In the article 'Mathematical Proof', Hardy (1929) refers to the ideas and positions of, among others, David Hilbert, L. E. J. Brouwer, Russell, Alfred Whitehead, and Wittgenstein. When Hardy assumes a Platonism, it is not as part of any implicit working philosophy of mathematics, but as a deliberate positioning.

In *A Mathematician's Apology*, one meets a deep concern about the possible roles of sciences as well as of mathematics, in particular in times of war. That science forms part of the war machinery was made evident by the First World War, and even more evident by the start of the Second World War. This is a deep preoccupation for Hardy. He sees war as an abominable phenomenon, and it is horrible for him of think of science as a resource for war technology. But what about mathematics? Should a mathematician feel responsible? Should a mathematician feel guilty? No doubt Hardy was troubled by such questions, but he states that 'a real mathematician has his conscience clear; there is nothing to be set against any value his work may have; mathematics is [...] a "harmless and innocent" occupation' (pp. 140–141).

This is the crucial claim in Hardy's conception of mathematics: we are dealing with a harmless and innocent occupation. However, Hardy is not talking about mathematics in general, but only about what he refers to as real mathematics.

Hardy's formulation could have been 'like a physicist, a chemist, and an applied mathematician, also a real mathematician has his conscience clean'. But Hardy does not want to say anything like this. Rather his claim is: 'in contrast to a physicist, a chemist, and many applied mathematicians, a real mathematician has his conscience clean'. He states:

> There is one comforting conclusion that is easy for a real mathematician. Real mathematics has no effects on war. No one has yet discovered any warlike purposes to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years. It is true that there are branches of applied mathematics, such as ballistics and aerodynamics, which have been developed deliberately for war and demand a quite elaborate technique: it is perhaps hard to call them 'trivial', but none of them has any claim to rank as 'real'. They are indeed repulsively ugly and intolerable dull: even Littlewood could not make ballistics respectable, and if he could not who can? (p. 140)

Then follows the conclusion as already quoted: 'A real mathematician has his conscience clear'.[2]

Rather than elaborating on the distinction between pure and applied mathematics, Hardy differentiates between real and trivial mathematics. According to Hardy, much mathematics is trivial, like school mathematics, calculus, and other such topics covered by introductory university textbooks, what can be referred to as engineering mathematics, and much applied mathematics. Contrary to real mathematics, such mathematics is 'trivial'. Hardy also finds it to be 'repulsively ugly' and 'intolerable dull'. These are very strong words that might reflect Hardy's profound aversion for the parts of mathematics, such as ballistics and aerodynamics, that are put into operation for purposes of warfare.

According to Hardy, mathematics developed as part of natural sciences can also be real, and he explicitly states: 'I count Maxwell and Einstein, Eddington and Dirac, among "real" mathematicians' (p. 131). He also states that he counts Isaac Newton as 'one of the world's three greatest mathematicians' (p. 71). As real mathematicians, they can

2 In a note, Hardy (1967, p. 152) makes some modifying observations with respect to §28 in the *Apology* (pp. 139–143) from where the quotations are taken. According to Hardy, the modifications are inspired by comments to the manuscript made by C. D. Broad and C. P. Snow. Hardy acknowledges that they might have some points and that he might have been too 'sentimental' in his formulations. However, he adds that he, anyway, decided not to make changes in this part of the manuscript. §28 is based on a short article that Hardy had published previously in 1940 in *Eureka*, the journal of the Cambridge Archimedean Society.

also have their consciences clear, while people contributing to trivial mathematics are not saved by Hardy.

I doubt that Hardy considered all elementary mathematics to be trivial. In the *Apology*, he refers to two classic mathematical proofs, one showing that the number $\sqrt{2}$ is irrational, and the other showing that there are infinitely many prime numbers.[3] We are dealing with elementary mathematical proofs, but I think that Hardy found them to be exemplars for making real mathematical discoveries. As mentioned, Hardy's use of the descriptor 'real' indicates that he embraces a Platonic view. Mathematical entities have a real existence, and properties of these entities become revealed through mathematical demonstrations. Thus, the two proofs that he refers to reveal the existence of non-rational numbers and the existence of infinitely many prime numbers. Together with Littlewood he tried to discover whether or not there exist infinitely many prime twins. When Hardy refers to real mathematics, he might well have in mind mathematics that contributes to revealing the properties of mathematical reality. He might think of trivial mathematics as not making such contributions, but operating within what already exists of mathematical entities. Trivial mathematics might combine techniques of huge complexities, it might provide a range of applications, but it does not contribute with mathematical discoveries.

One way of cleaning a mathematician's conscience could be to show that what is done through mathematics can be only 'good things'. Such a line of argumentation could take an almost religious format. For instance, Newton was a devoted believer in God. He revealed how mathematics captures the laws of nature, and therefore the way God had created the world. Mathematics could be thought of as an expression of the rationality of God, and as a consequence, one cannot say anything other than good things about mathematics. Versions of this line of thought have been repeated again and again. But not by Hardy. Any religious flavouring of an apology for mathematics was impossible to him.

Hardy cleans the real mathematicians' consciences by claiming that what they are doing is without any use. While trivial mathematics can be useful, there is no usefulness to be associated to real mathematics: 'I have never done anything "useful". No discovery of mine has made, or

3 In Chapter 7 of this volume, the proof for the infinity of prime numbers is presented and discussed with reference to intuitionism.

is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world' (p. 150). Contrary to trivial mathematics, real mathematics does not make a difference, neither for the good nor for the bad. Real mathematics has no utilitarian value whatsoever, whether in times of peace or in times of war. Real mathematics is harmless and innocent.

Hardy uses different words to express that real mathematics is harmless and innocent, as, for instance, 'gentle' and 'clean'. After making some references to Gauss, he states the following:

> If the theory of numbers could be employed for any practically and obviously honourable purpose, if it could be turned directly to the furtherance of human happiness or the relief of human suffering, as physiology and even chemistry can, then surely neither Gauss nor any other mathematicians would have been so foolish as to decry or regret such applications. But science works for evil as well as for good (and particular, of course, in time of war); and both Gauss and lesser mathematicians may be justifying in rejoicing that there is one science at any rate, and that is their own, whose very remoteness from ordinary human activities should keep it gentle and clean (pp. 120–121).

Hardy does not propose any theory about the neutrality of science. In fact, he highlights the opposite, that 'science works for evil as well as for good', and that it does so, in particular, 'in times of war'. According to Hardy, mathematics is not any neutral science. The only thing he insists upon is that real mathematics operates beyond any evil-good controversies. Not because it contains any intrinsic goodness or any sublime ethical qualities, but because it is useless.

His utilitarian perspective on mathematics is consequential, as utilitarianism as an ethical position provides a non-religious perspective on ethical questions. Whether something is 'good' or 'bad' cannot be judged according to some sublime ethical or religious principles, but only with respect to its utilitarian implications. By stipulating that real mathematics has no such implications, Hardy saves this part of mathematics from being considered harmful in any way. It is simply innocent.

Why then work with real mathematics? As Hardy set aside any possibility for providing a utilitarian justification for such work, one needs to ask what kind of justification is then possible. Hardy is well aware that he leaves only a narrow space for himself to articulate justifications. His reaction, however, seems to be that he, in fact, does not need much such space. In some formulations in the *Apology*, he uses the notion of being 'serious'. Chess problems might be extremely challenging, but, according to Hardy, compared to real mathematical problems they are unimportant: 'The best mathematics is *serious* as well as beautiful – "important" if you like, but the word is very ambiguous, and "serious" expresses what I mean much better' (p. 89, italics in original).

Elaborating justifications for working with real mathematics does not appear necessary to Hardy. To ask for any such justification, utilitarian or not, is like asking Mozart to provide a justification, utilitarian or not, for making his compositions. Hardy would rather state that Mozart's work is serious (and innocent) like any other work of art, including real mathematics.

A doctrine

In *A Mathematician's Apology*, Hardy (1967) elaborates a thesis of innocence, which can be summarised in the following way: sciences might work for the evil as well as for the good. Within science, however, there exists a small domain that is not under suspicion for being harmful, and this is real mathematics. It is useless, and as such it is harmless and innocent.

Hardy's thesis of innocence can be related to, but also contrasted with, a doctrine of neutrality. While Hardy's thesis is well-articulated and refers to a particular domain within mathematics, the doctrine of neutrality often operates as a discursive pattern and includes any kind of mathematics. The doctrine is deployed whenever one wants to cut off a discussion of possible socio-political impacts of mathematics. It turns into an ideology by assuming that mathematics as such is harmless and innocent, and that one can conduct mathematics research without engaging in critical reflections about what might be done through mathematics. Contrary to Hardy's thesis of innocence, the doctrine of neutrality operates as a discursive given, and not as a claim in need of justification. The doctrine is part of an implicit working philosophy of mathematics. It is called into operation when socio-political issues are stipulated as irrelevant when doing mathematical research.

I see the doctrine of neutrality as a disproportionate and exaggerated shadow of Hardy's thesis of innocence. The doctrine concerns any kind of mathematics, so that mathematics as such becomes stipulated as being neutral. There is no need to make specifications with respect to mathematical topics: algebra, calculus, number theory – all such subjects are harmless and innocent. They are neutral. Nor is it necessary to make specifications with respect to levels of mathematics: elementary mathematics, advanced mathematics, research mathematics – all are neutral subjects.

The doctrine of neutrality is materialised in the organisation of university studies in mathematics. Naturally there is a variety of such study programmes, but what I have in mind here I refer to as the university mathematics tradition.[4] This tradition includes the following characteristics: (1) It defines the curriculum in well specified units such as Calculus 1, Calculus 2, Linear Algebra, Algebra 1, Algebra 2, Probability Theory, non-Euclidean Geometry, Projective Geometry, and so on. (2) Among these units there is no space for a philosophy of mathematics including ethical discussions related to the use of mathematics, and not much space for history of mathematics, which could include socio-political reflections. (3) Within the units, ethical and socio-political controversies that could be related to the particular mathematical subdiscipline are not addressed. (4) All tests and exams focus on mathematical competencies.

When we are dealing with a doctrine, the structure of its justification need not be explicit, nor even coherent. A doctrine is a general positioning, which can be articulated in different contexts and make part of a variety of discourses, insisting that mathematics is detached from socio-political issues. *A Mathematician's Apology* has turned into a most questionable publication, as it has enabled many to maintain a doctrine of neutrality as part of a working philosophy of mathematics. The doctrine leads to a conception of mathematics that fosters a banality of mathematical expertise (see Skovsmose, 2020). This banality embraces the ignorance of possible implications of what one is doing. It ignores the context within which mathematical research is conducted and where mathematics is brought in action.

4 One can find a characterisation of the school mathematics tradition in Skovsmose and Penteado (2016).

A sigh

Hardy (1967) did not elaborate his thesis of innocence starting from systematic philosophical observations, but rather from his experiences as a mathematician. As already referred to, his principal justification for the thesis built on observations such as: 'No one has yet discovered any warlike purposes to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years' (p. 140). The justification for his thesis is empirical, referring to what can be observed, or rather, to what has not (yet) been observed.

It is not surprising that Hardy refers to number theory, which is his paradigmatic case of real mathematics. That he also refers to relativity is a surprise to me. Relativity theory provides a mathematical conception of nature, and Hardy thinks of Albert Einstein, and other great physicists, as contributing to the domain of real mathematics. Hardy states: 'The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present at any rate, almost as useless as the theory of numbers' (pp. 131–132). According to Hardy, such examples of applied mathematics are not trivial, but real.

However, already in 1940 when Hardy published this statement, it was possible, with developments in the theory of relativity, to conceptualise the possibility of an atomic bomb, as expressed dramatically in the equation $E = mc^2$. However, the route from this theoretical insight to the actual construction of a bomb only became identified in steps, many of which were kept as military secrets. In 1945, with the destruction of Hiroshima and Nagasaki, it was demonstrated to everybody that the theory of relativity was implicated as an integral part of modern war machinery. Hardy's justification of the thesis of innocence by referring to relativity is simply wrong.

What would Hardy make of this? He witnessed the conclusion of the Second World War and the destructions of Hiroshima and Nagasaki, but I am not aware he tried to make any revision of this formulation in his *Apology*. I also think that he did not really think of his remark about relativity as being that crucial for his justification. The remark appears as an aside from his principal argument referring to number theory.

In the article 'Formatting Power of "Mathematics in a Package": A Challenge for Social Theorising?', Keiko Yasukawa and I (2009) discuss modern cryptography.[5] Cryptography has a long history, and was applied already in Antiquity. The development of cryptography can be directly related to technological developments, and different mechanical machineries for coding and decoding have been invented, reaching an extreme sophistication during the Second World War.

Two years after the death of Hardy, Claude E. Shannon (1949) published the article 'Communication Theory of Secrecy Systems', which establishes the opening for an advanced mathematical approach. In *New Directions in Cryptography*, this approach was elaborated in detail by Whitfield Diffie and Martin Hellman (1976). As Yasukawa and I point out, the very identification of these new directions in cryptography is based on profound number theoretical insights. The idea is to construct a technique for encoding and decoding that can be handled automatically without compromising security measures. The computer makes such automatisation possible, so that huge amounts of data can be encoded and decoded. The whole process is complex, but the principal observation, which ensures the safety of the whole approach, is related to a simple observation: breaking the code turns out to be equivalent to being able to factorise a number that is the product of two huge prime numbers.

In *Number Theory in Science and Communication*, Manfred R. Schroeder (1997) states that if we are dealing with a 200-digit number that is the product of two prime numbers of more or less equal size, the factorisation cannot be completed within any conceivable time limit. He points out that 'not so long ago, the most efficient factorising algorithms on a very fast computer were estimated to take 40 trillion years, or 2000 times the present age of the universe' (p. 131). Certainly, this statement is time-dependent. The quotation here is from the third edition of the book, while in a previous edition from 1983, Schroeder makes the same statement, but referring to a 100-digit number. Newer editions of the book have been published, but I have not yet had the opportunity to check the possible reformulations of the quoted statement. Certainly new algorithms can be identified, and computers are becoming more and

5 See also Yasukawa, Skovsmose, and Ravn (2012).

more powerful. That we, independent of such development, continue to face a task that cannot be completed within any conceivable time limit is crucial for the whole cryptographic approach. To make such a claim about factorisation depends on a deep number theoretical insight.

How could it be that a factorisation explodes in complexity? Here comes a number n with 200 digits:

783350087122241855766633268290884426110902343377681777777325699
400967226183766454225112566700999599333851557363299228890099238
238882812482500038888217888299994898921156667211390080900765334
87112387111

I just pressed number keys 200 times, so if this number n turns out to be a product of two prime numbers of more or less the same size, it would be a most unlikely coincidence. However, if so, we should expect that it would take trillions of years until we discover which two prime numbers we are dealing with. My intuition does not point towards such a conclusion. To me the number looks large, but that the equation $n = p^1 p^2$ represents such overwhelming computational complexities, I could never imagine. In order to reach such an insight, one needs to draw on profound number theoretical insights. Furthermore, new number theoretical insights concerning the distribution of prime numbers and efficient algorithms for factoring might lead to a modification, if not a direct falsification, of the claim.

As cryptography makes an indispensable part of modern war technology, number theory forfeits all claim to be harmless and innocent. This observation is devastating for Hardy's justification of the thesis of innocence. Number theory turns out to be extremely useful, in particular in times of war. It can be harmful in just the same way as ballistics and aerodynamics can be.

I imagine that Hardy is sitting in the same comfortable chair as shown at the cover of my edition of the *Apology*. He has an attentive look, his glasses are a bit down his nose, his hands are empty, and he seems just ready to grasp a book or a paper. I imagine that he gets an opportunity to look at the paper by Yasukawa and me. I have no doubt that, after a few moments, he will put aside the paper and ask for the original references. After looking through them, his expression will change. He is not really looking at anything or at anybody anymore.

His look turns inwards. I have no idea what he is going to say, but I can imagine his deep sigh.

While the maintenance of an articulated thesis might depend on the status of its justification, a doctrine could easily survive even the most downright falsification. Although Hardy's justification of the thesis of innocence has collapsed, its disproportionately exaggerated shadow, the doctrine of neutrality, continues to be seen everywhere. This shadow provides a cover for mathematical research and university studies in mathematics to maintain a profound silence with respect to ethical and socio-political issues.[6] Mathematics continues to be conceptualised as harmless and innocent. But it is not.

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6 In line with such an observation, Hacking (2014) makes the following comment: 'G. H. Hardy's fantasy about pure mathematics, *A Mathematician's Apology*, did much harm to philosophers' (p. 185).

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