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BREAKING IMAGES

ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

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4. Formalism, structuralism, and the doctrine of neutrality

Ole Skovsmose

The doctrine of neutrality states that mathematics can be researched and developed without considering any ethical or socio-political issues. This doctrine became elaborated and argued in detail by the school of logical positivism. By the turn of the nineteenth century, a range of paradoxes and inexplicable mathematical phenomena appeared, a situation referred to as the foundational crisis of mathematics. To many, intuition was the scoundrel, and it had to be eliminated from mathematics. Formalism provided a principal approach by identifying mathematics with formal structures. This idea was embraced by logical positivists who claimed that mathematics as the language of science ensures the ethical neutrality of science. They considered mathematics not only as being neutral itself, but also as a guarantee for scientific neutrality in general. In this way, a most profound stipulation of the doctrine of neutrality was reached. Formalism developed into structuralism, which described mathematics as an architecture of pure formal structures. As part of the structuralist conception of mathematics, the doctrine of neutrality was expanded from being a conception of mathematical research to become also a doctrine shaping educational practices in mathematics. I am going to confront this conception. The doctrine of neutrality is a stipulation, which makes us ignore that a profound politicisation of both mathematics and mathematics education might be taking place.

Introduction

On the 22 June 1936, Moritz Schlick was murdered on the broad steps of the main entrance of the University of Vienna. He was shot from close range by a former student, Johann Nelböck. Schlick died immediately.

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In 1922, Schlick had been nominated as professor in *Naturphilosophie* (philosophy of nature) at the university, and Nelböck had studied there with Schlick as his advisor. In 1931, Nelböck graduated as a doctor in philosophy. During the trial, he gave different explanations for the killing, one referring to jealousy. He also claimed that Schlick's anti-metaphysical philosophy had troubled him. Nelböck was sentenced to ten years in jail.

The event became a controversial issue with much public attention. Although Schlick was a German Protestant, he became portrayed in the press as a key figure in suspicious academic Jewish circles, and Nelböck became celebrated by Nazi movements. In 1938, after the *Anschluss*, the German annexation of Austrian, Nelböck was released.

Soon after his nomination, Schlick organised a discussion group that later became known as the Vienna Circle (*Wiener Kreis*). This circle formulated a view on science and mathematics that came to be known as logical positivism. The circle was deeply engaged in recent developments in science, mathematics, and logic. They were conversant with developments in physics. They studied carefully the monumental work *Principia Mathematica* by Alfred Whitehead and Bertrand Russell, published in three volumes in 1910–1913. The *Tractatus Logico-Philosophicus* by Ludwig Wittgenstein, published in 1922, was also studied with extreme intensity.

The circle was also deeply engaged in the recent political developments in Austria and Germany. They launched a strong attack on any form of metaphysical thinking, including Nazi ideologies. No doubt, Schlick's anti-metaphysical philosophy troubled not only Nelböck, but many from the Nazi movement.

The anti-metaphysical philosophy initiated by the Vienna Circle ended up providing a broad platform for claiming that science and mathematics can be kept separated from socio-political and ethical issues, that they are neutral subjects. In the previous Chapter 3, we saw how Hardy's thesis of mathematics being innocent turned into a dogmatic claim of mathematics being neutral. In this chapter, we are investigating a much broader philosophical trend, which establishes a formidable manifestation of this dogmatism.[1]

1 This dogmatism was confronted by a critical conception of mathematics that I return to in Chapter 11 in this book.

As an initial step into this dogmatism, I refer to the *elimination of intuition* from mathematics, which represents a de-contextualisation of mathematical notions and reasonings. I refer to *formalism*, which emerges as the result of this elimination of intuition, and which identifies mathematics with formal structures. Then I provide an exposition of *logical positivism* that embraces formalism and claims that not only mathematics but science in general is neutral and detached from any socio-political stance. After that, I present *structuralism*, which represents a particularly elaborated version of formalism also embracing the dogma of neutrality. I show how the Modern Mathematics Movement emerged as an educational version of structuralism and manifests the de-policisation of mathematics education. As a kind of epilogue, I make a few comments about 'Poor Piaget!'.

Elimination of intuition

The elimination of intuition from mathematics includes three components: making *explicit* hidden axioms and assumptions that in fact are used in developing a mathematical theory; eliminating *ontological* issues referring to assumptions about the nature of mathematical objects; and *identifying* and formalising the logical patterns of deduction and reasoning used in mathematics.

Since Antiquity, Euclid's *Elements* has been taken as the paradigmatic case of mathematical deduction.[2] A deduction must start out from something, and this 'something' was presented by Euclid in terms of five axioms (which by Euclid were referred to as postulates). From these axioms, logical deduction leads to a range of theorems. If the axioms were true, all theorems would be true as well, since logical deduction provides a delivery of truth. As Euclid's axioms appeared simple (although the fifth axiom seemed less so), their truth could be grasped immediately by intuition, and due to the strict deductive organisation of the *Elements*, all theorems could be considered true as well. So it was generally assumed.

This perception of the *Elements* only became challenged during the nineteenth century, where different studies revealed that more than the

2 See Joyce (1998).

five axioms were involved in the deduction of theorems. Euclid had also applied intuition, for instance concerning space. This came as a shock: How could it be that this had been overlooked for more than 2000 years? In 1882, Moritz Pasch (1912) prepared the foundations for an extended axiomatics for Euclidean geometry. The process of making all applied axioms explicit was brought together in *Foundations of Geometry* (Hilbert, 1950), the first German version of which appeared in 1899. While the *Elements* includes five axioms, the *Foundations of Geometry* builds on twenty-one axioms (later it was showed that one of them was redundant). Besides the five included in the *Elements*, one also finds, for example, the axiom:

> Let A, B, C be three points not lying in the same line and let *a* be a line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the line *a* passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC. (Hilbert, 1950, p. 4)

This axiom, first made explicit by Pasch, was applied by Euclid, but just as an intuitive insight. It was taken as a given that, when a straight line cuts one of the sides of a triangle, it will also cut one of the other sides (except from the situation where the line passes through a vertex of the triangle). The axiom states that when a straight line cuts into a triangle, it will not disappear in the interior of the triangle, but cut out of the triangle as well. There are no Bermuda triangles to be found in Euclidean geometry.

In order to eliminate intuition from mathematics, any deduction should be based on explicitly stated axioms. This was exactly what was prepared for by Pasch and accomplished by David Hilbert with respect to Euclidean geometry.

Ontological issues have been an ongoing challenge for the philosophy of mathematics: What is a number, a point, a line, or any other 'mathematical object' for that matter? In Euclid's *Elements*, a point becomes defined as that which cannot be divided. This sounds clear enough, but it appears unclear what is the point of making this definition. It is not used later on in the deductive processes. Maybe the clarification of ontological issues is not crucial for mathematical proving. This point was clearly recognised by Hilbert (1950), who initiates the presentation in *Foundations of Geometry* by stating:

> Let us consider three distinct systems of things. The things composing the first system, we will call *points* and designate them by the letters A, B, C [...], those of the second, we will call *straight lines* and designate them by the letters a, b, c, [...], and those of the third system, we will call *planes* and designate them by the Greek letters α, β, γ, [...]. The points are called the *elements of linear geometry*; the points and straight lines, the *elements of plane geometry*; and the points, lines, and planes, the *elements of the geometry of space* or the *elements of space*. (p. 2, italics in original)

Hilbert's point is that, for presenting geometry in an axiomatic format, the very nature of its objects is irrelevant. They can be referred to as 'things'. In the quotation, he refers to 'points', 'lines', and 'planes', but, as he in 1891 had pointed out in a conversation with two other mathematicians at a train station, he could as well have referred to 'tables', 'chairs', and 'beer mugs'.[3] Hilbert did highlight this point in 1891. For a geometric theory, the intrinsic qualities of points, lines, and planes are irrelevant; only their relationships are relevant. Such relationships, and nothing else, become specified through the axioms of geometry. In this way, Hilbert tried to eliminate intuition from ontological considerations, simply by considering ontology superfluous.

If an elimination of intuition from mathematics reasoning is to be properly carried out, one needs a firm grasp of what logical deduction could mean. This was the principal idea of Gottlob Frege's life project. He wanted to provide an enumeration of all valid forms of logical deduction. But how to do this? It would become a long list, and in what order should it be organised? Frege had a clear approach in mind; he wanted to organise all valid logical deductions in an axiomatic system, and in the *Begriffsschrift* (Frege, 1967), the first German version of which was published in 1879, he presented how this could be done. His *Begriffsschrift* provides a start of the formulation of modern formal logic.

Frege presents a set of logical axioms, seven in total, and two specific rules of inference, claiming that this defines a system that has as theorems precisely all valid forms of logical deduction. The system concerns propositional logic, but on top of this Frege added predicate calculus. Here, however, let me concentrate on its propositional basis.

Frege used a particular formal terminology, which did not become common. However, his presentation in the *Begriffsschrift* has been

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3 See Shapiro (2000, p. 151), and Hilbert (1935, p. 403).

carefully reworked in a symbolism that is now common in modern logic. A huge effort was made in *Principia Mathematica* by Alfred Whitehead and Bertrand Russell (1910–1913). The axioms that Whitehead and Russell used are a bit different from those suggested by Frege, but the scope of the axiomatic systems are quite the same. For the propositional calculus, Whitehead and Russell operated with the following five axioms:

1. $(p \lor p) \Rightarrow p$
2. $q \Rightarrow (p \lor q)$
3. $(p \lor q) \Rightarrow (q \lor p)$
4. $(p \lor (q \lor r)) \Rightarrow (q \lor (p \lor r))$
5. $(q \Rightarrow r) \Rightarrow ((p \lor q) \Rightarrow (p \lor r))$

Russell and Whitehead maintained the two rules of inference as formulated by Frege. The first states that if A⇒B *is a theorem or an axiom, and A is a theorem or an axiom, then B is a theorem*. The second states that one can substitute a symbol with another. If, for instance, one has proved the formula $p \Rightarrow (p \lor p)$, then one can also conclude that $q \Rightarrow (q \lor q)$. No other rules of inference than these two were applied.

In summary, the elimination of intuition from mathematics was carried out by making hidden axioms explicit, by eliminating ontological issues from mathematical theorising, and by capturing mathematical reasoning by formal axiomatic systems. This triple-strategy for eliminating intuition created a new way of looking at mathematics. The triple strategy became defining for the formalist outlook.

Formalism

Mathematics has been seen as a unique way of obtaining certainty. When a mathematical proof has been completed, we conclude that the proved theorem is true, and true with certainty. While doubt and uncertainly accompany many forms of human knowledge, it can apparently be eliminated from the domain of mathematics. Mathematics seems a fortification against any possible invasions of scepticism. Therefore, it came as a shock when the fortification seemed to be collapsing.[4]

4 In my discussion of the foundational crisis of mathematics I draw on Ravn and Skovsmose (2019).

By the turn of the nineteenth century, a range of paradoxes and inexplicable mathematical phenomena appeared, creating a situation referred to as the *foundational crisis of mathematics*. In 1901, one paradox was discovered by Bertrand Russell; it was also identified by Ernst Zermelo in 1899, but he communicated it only to a small circle of colleagues from Göttingen University, including Hilbert. The paradox has the following form: Let M denote the set of sets that are not members of themselves, thus $M = \{x \mid x \notin x\}$. Then let us ask: is M a member of itself? If the set M is a member of itself, it has the property $M \notin M$. If M is not a member of itself, it has the property $\neg(M \notin M)$. In other words, we can conclude $M \in M$. Thus, we have $M \in M$ if and only if $M \notin M$.

Not only did such explicit paradoxes emerge, but also strange phenomena were observed. Georg Cantor (1874) presented a new understanding of the notion of set, which until then had been taken as an uncomplicated intuitive notion. He showed that the infinity of real numbers is of a higher degree than the infinity of natural numbers. In fact, he revealed the existence of an infinity of degrees of infinities. Guiseppe Peano (1890) discovered a curve, commonly referred to as the Peano curve, which is a surjective and continuous function from the unit line the unit square. This curve, with the surprising property of being able of cover an area, has also been referred to as 'the bald man's hope'. If just one hair, long enough, is left, then the baldness can be properly covered.

How could it be that mathematics, which appeared so carefully elaborated through proofs and theorems, could run into logical contradictions? What did the occurrence of new strange mathematical objects signify? Something seemed to have gone wrong. But how?

Logicism, formalism, and intuitionism represent three main approaches for addressing the foundational crisis. To logicism and formalism, the scoundrel was intuition, and the elimination of intuition from mathematics forms part of these two approaches. Contrary to these positions, intuitionism claims that intuition is crucial to mathematics, and that the paradoxes emergence when mathematics procedures become led astray by formalist procedures.[5] In the following we concentrate on how Hilbert addressed the crisis.

5 In Chapter 7 in this volume, I discuss more carefully the intuitionist approach to mathematics.

Hilbert suggested a two-step metamathematical programme inviting a formalist outlook on mathematics. First, mathematical theories had to be formalised. This could be done by squeezing every juicy drop of intuition out of mathematics. Then only formal structures would remain. Second, these formal representations of mathematical theories had to be investigated, in particular with respect to completeness and consistency. If the completeness and consistency could be proved, then mathematical theories would be vaccinated against paradoxes.[6]

However, in 1931 Hilbert's programme suffered a knock-out, when Kurt Gödel (1962) published his famous incompleteness theorem. This theorem states that if a formal system of a certain complexity is consistent, it will be incomplete. The idea of representing mathematical theories by complete and simultaneously consistent formal systems was revealed as an illusion. Gödel's proof presupposes that the formal system in question is rich enough to include an axiomatisation of standard arithmetic, which was a minimal requirement for the whole metamathematical programme.[7]

The original idea of metamathematics was that mathematical theories could be *represented* by formal systems. Soon emerged the idea that mathematical theories could be *identified* with formal systems. This idea acquired much force, even after the metamathematical programme had stumbled over Gödel's incompleteness theorem. The claim of identity between mathematics and formal structures is defining for formalism as a philosophy of mathematics. Hilbert has often been referred to as the father of formalism, but I doubt if he thought of formal systems as being anything more than representations of mathematical theories.

Formalism appears a powerful position, as it provides straightforward answers to such classic philosophical questions as: What is mathematics? The question simply becomes identical to the question: What is a formal system? This later question can be answered in specific steps by clarifying the notions of *alphabet*, *formula*, *axiom*, *rule of inference*, *proof,* and *theorem*.

A formal system has to operate with an *alphabet,* which refers to the set of symbols that can be applied. Such an alphabet can include symbols such as p, q, r, (,), $\vee$, $\Rightarrow$, $\forall$, $\in$, and $\exists$. For any specific formal system, the

6 For a detailed presentation of metamathematics, see Kleene (1971).
7 For further discussions of Gödel's incompleteness theorem, see Budiansky (2021) and Goldstein (2005).

list of allowed symbols must be explicitly enumerated. It should be well-defined whether or not a symbol belongs to the alphabet or not. It needs to be specified which sequences of symbols count as formulas in the system. One can think of this definition as the grammar of the formal system. A grammar could, for instance, define the sequence $(p \vee q)$ as being correct, and the sequence $(\Rightarrow p \vee q$ as being incorrect. The whole grammar has to be formulated in such a way that it is well-defined whether or not a sequence of symbols is a formula or not.

Some formulas have to be enumerated as *axiom*s. This set will serve as a departure for the deductions to be made. Naturally, there are many issues related to the selection of axioms, as, for instance, not selecting axioms that might lead to contradictions. The *rules of inference* that are going to be applied in the system have to be enumerated. Such rules specify how one, from one or more formulas, can derive other formulas. The basic idea is that *if* the original formulas (the premises) are true, *then* the derived formulas (the conclusions) will be true as well. No formal system demonstrates the actual truth of any theorems, but it shows what can be considered true if the axioms are considered true. In a formal system, the notion of truth is of a hypothetical if-then nature.

A *proof* can be defined as a sequence of the formulas $F_1, \ldots, F_n$, where any formula F_i (where $1 \leq i \leq n$) is either an axiom or can be derived from one or more of the formulas in the sequence $F_1, \ldots, F_{i-1}$ in accordance with the rules of inference. This definition of proof brings us to the definition of a *theorem* as a formula which occurs as the last formula F_n in a sequence of formulas $F_1, \ldots, F_n$, that composes a proof.

By such a clarification of alphabet, formula, axiom, rule of inference, proof, and theorem, one gets a definition of formals system, and, as a consequence, a definition of mathematics according to formalism.

Logical positivism

The formalist interpretation of mathematics had a huge impact on the formulation of logical positivism as a philosophy of science in general. According to logical positivism, mathematics and scientific theories have to be kept strictly detached, not only from intuition, but from any form of contextualisation. They have to be kept separated from subjective preferences, religious convictions, ethical principles, cultural traditions, political priorities, and from any form of metaphysical thinking.

In *A Mathematician's Apology*, Hardy formulated a thesis of innocence, with respect to what he referred to as 'real' mathematics. This thesis, however, invoked the much broader dogma of neutrality, according to which any form of mathematics can be researched and developed separately from ethical and political considerations. Logical positivism establishes an even much broader dogma of neutrality according to which not only mathematics, but also science in general, can be kept, and must be kept, ethically and politically neutral. This dogma came to dominate the perspective on mathematics and science, and was rarely questioned until the late 1960s, when critical conceptions of mathematics and of sciences were formulated.[8]

The Vienna Circle, as organised by Moritz Schlick (1886–1936), included philosophers, scientists, and mathematicians. Rudolf Carnap (1891–1970), Herbert Feigl (1902–1988), Kurt Gödel (1906–1978), Hans Hahn (1879–1934), Otto Neurath (1882–1945), and Friedrich Waismann (1896–1959) were among them.[9] The Circle was deeply engaged in actual developments in science and mathematics. They studied Albert Einstein's formulation of the theory of relativity, and the principles of quantum mechanics. Hilbert's metamathematical programme was carefully discussed, and recent developments in formal logic were investigated. In 1929, Gödel completed his PhD studies in formal logic with Hahn as his supervision,[10] and two years later he presented his famous incompleteness theorem. Wittgenstein's *Tractatus*, published in 1922 in a German-English parallel edition, was studied carefully by the Circle. It provided a principal inspiration for formulating the

8 See Chapter 11 in this volume for the formulation of a critical conception of mathematics.

9 Several other people became associated with the Vienna Circle, for instance Hans Reichenbach, who worked in Berlin. Together with Carnap, he edited the journal *Erkentnis* (*Knowledge*) that expressed the outlook of logical positivism. Carl Hempel also worked in Berlin. Karl Popper was around, but even though he was actively contributing to the discussion of science and shared many of the concerns of the Vienna Circle, he was never invited by Schlick to join.

10 The result of this study is found in Gödel (1967). By proving the completeness theorem of predicate logic, Gödel demonstrated that Frege's intuition was sound; the axiom system that he presented in the *Begriffsschrift* as the foundation of predicate logic was in fact complete.

overall position of logical positivism including the dogma of neutrality. Wittgenstein was also invited by Schlick to join meetings of the Circle.[11]

Members of the Vienna Circle were deeply concerned about political developments including the growing anti-Semitism. For them, Nazi conceptions such as 'Arian Physics' or 'degenerate Jewish physics' had nothing to do with science; as meaningless metaphysical notions, they had to be eliminated from any scientific outlook. Looking more carefully at scientific theories, one might find a broader range of metaphysical assumptions and preconceptions, not only of political but also of philosophical, religious, and psychological nature. According to the Vienna Circle, all such features of metaphysics had to be eliminated from science. They found that they were facing a huge task in a most difficult period of time, namely to clean up science and to ensure that it got its proper neutral format.[12]

In an attempt to eliminate metaphysics from the domain of science, the Vienna Circle formulated the principle of verification. According to this principle, a statement has a meaning if and only if it is possible to specify some observations that can serve as empirical evidence for that statement. If such a specification is not possible, the statement is meaningless. As an example, we can take the statement 'God is almighty.' As one cannot point out any possible empirical observations that could support this statement, it is meaningless. In general, religious claims end up in the waste bin together with any other forms of supposed nonsense. So do many statements from psychology and psychoanalysis. The waste bin also becomes stuffed with ethical statements, as no empirical evidence for such statements can be identified. Furthermore, established disciplines such as physics need critical investigations since, for instance, the concept of force might include metaphysical features.[13]

A variety of specific formulations of the principle of verification was carefully investigated by the Vienna Circle. However, it turned out that whatever formulation one gave the principle, one could not escape the dilemma that either the formulation would be too loose, meaning

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11 For a careful study of the Vienna Circle, see Stadler (2015). For captivating presentations of the Vienna Circle, see Edmonds (2020), and Sigmund (2017).

12 Carnap (1959, first published in *Erkenntnis* in 1932) made a powerful presentation of this cleaning programme.

13 See Jammer (1957).

that obvious metaphysical statements came to count as meaningful, or it would be too tight, meaning that general natural laws of physics became relegated as meaningless. Furthermore, what about the very principle itself? How could you verify the principle of verification? As it appears impossible to specify what empirical evidence might support the principle, it seems itself to become meaningless.[14]

The approach to eliminate metaphysics, however, also followed another much more powerful departure. Logical positivism expressed a huge doubt with respect to natural language. In doing so, it drew directly on the formalist conception of mathematics. The grammar of natural language was all too loose, making ample space for formulating any kind of statement, also with a profound metaphysical content. Natural language opens an extensive space for expressing nonsense in grammatically correct ways. When used as the language of science, natural language must be under suspicion.[15]

In the *Tractatus*, Wittgenstein (1992) assigned a particular role to formal language. Here he consistently talks about language as singular, and it really has to be read as *the* language. This is the formal language from *Principia Mathematica*, and the language that formalism had cultivated. This language Wittgenstein sees as *the language of science*, emphasised throughout the *Tractatus* and brought together in §6 and §7. In the concluding paragraph of the *Tractatus*, one can read:[16]

> §7 Whereof we cannot speak about, thereof one must be silent.

Many times, this paragraph has been read as an elegant and artistic conclusion of the book, but it is much more than that. It condenses Wittgenstein's whole conception of science and ethics. §7 has to be read together with §6. While §7 states what *cannot* be said, §6 states what *can* be said:

> §6 The general form of a truth-function is: $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$. This is the general form of proposition.

14 An overview of the discussion of the principle of verification is presented by Hempel (1959), first published in 1950.

15 One important contribution to the critique of natural language was previously formulated by Russell (1905) in the article 'On Denoting'.

16 There exist several translations of the *Tractatus* into English. Here I cite the first translation by C. K. Ogden from the original German-English edition of 1922.

By a truth-function, Wittgenstein refers to a property of a composed proposition, namely that its truth value is determined by the truth values of the propositions of which it is composed. The expression $[\bar{p}, \bar{\xi}, N(\bar{\xi})]$ is Wittgenstein's shorthand for an arbitrary proposition built up by logical connectives.[17] Wittgenstein claims that any proposition has this form. If a linguistic formulation does not have the property of being a truth-function, it must, according to §7, be passed over in silence. Wittgenstein's claim is that the language of science is truth-functional.[18]

This claim was carefully discussed by the Vienna Circle. In the *Logical Syntax of Language*, the original German version of which was published in 1934, Carnap (1937) elaborated on the claim that a formal language can serve as the language of science. The discussion is rich in details, and Carnap recognised that one needs to apply a version of formal logic with a higher degree of complexity than the one Wittgenstein referred to in the *Tractatus*. I see Carnap's discussion in *Logical Syntax of Language* as a careful elaboration of the clue provided by Wittgenstein in §6 of *Tractatus*.

What now to think of that which cannot be expressed in formal language? As mentioned, Wittgenstein's answer comes in §7: Remain silent! Logical positivism agrees: Science has to concentrate on what can be expressed in formal language, and to leave the rest aside. Thus, §7 is a condensed expression of the claim that no metaphysical elements can be part of science, whether they take the form of religious convictions, political positions, or ethical principles. Together §6 and §7 provide as condensed formulation of the total separation between science and any value statements.

17 The connectives could be and , as used by Whitehead and Russell in *Principia Mathematica*. In 1913, Henry Sheffer showed that it is possible to define the other connectives from just one connective, now referred to as the Sheffer stroke. Informally, the Sheffer stroke can be defined as 'not both', meaning that it occurs as a negation of a conjunction. In Wittgenstein's symbolism, the Sheffer stroke is referred by be the symbol . Thus by Wittgenstein refers to an arbitrary truth-function expressed by means of the Sheffer stroke.

18 The formulation seems to indicate that Wittgenstein thinks of the language of science as being that of proposition logic. That is certainly a simplistic claim. As a minimum, one should think of the language of predicate calculus as a prerequisite for the language of science. Any formulation of laws of nature would presuppose such a language.

Acknowledging the principal idea of logicism that mathematics and logic are the same, logical positivism claims that mathematics as the language of science ensures that science does not include metaphysical elements and that science turns ethically neutral. In this way we have reached a most profound legitimation of the doctrine of neutrality.

Due to their explicit anti-Nazi positions and, in several cases, also to their Jewish origins, many members of the Vienna Circle left Austria after Adolf Hitler came to power. Many escaped to the United States of America.[19] As part of the transplantation into an English-speaking context, logical positivism did change.[20] From being a critical stance with respect to the present state of science, including a profound critique of Nazi ideologies, it turned into a devise for legitimising science as, in fact, being detached from socio-political issues. In this way, the transplanted version of logical positivism came to operate as a legitimation of what was taking place in most university studies in sciences and mathematics, not only in the USA but the world over. Logical positivism turned into a legitimation of not engaging in socio-political issues as an integral part of any such study programmes. From providing a departure for a critique of science, logical positivism turned into a broadly assumed convenient dogma about neutrality.

Structuralism

Structuralism can be interpreted as an elaborated version of formalism, and structuralism had a profound impact on mathematical research. It

19 In 1935, Carnap emigrated to the USA. Feigl's parents were not religious, but they were Jewish, and in 1931 he left for the USA. In 1939, Gödel got a position at the Institute of Advanced Studies in Princeton, with which also Einstein was associated. In 1934, Neurath fled to the Netherlands and later on to England. Waismann was a Jew, and in 1938 he emigrated to the USA. In 1933, Reichenbach was dismissed from his work due to his Jewish background, and in 1938 he moved to the USA. Hempel's wife was of Jewish origin, and in 1937 they emigrated to the USA. Popper had Jewish origins as well, and in 1937 he emigrated to New Zealand.

20 The first English introduction to the ideas of the Vienna Circle was presented by Ayer (1970), when he published *Language, Truth and Logic* in 1936. Other presentations in English are found in Ayer (1959), Carnap (1962), Hempel (1970), and Reichenbach (1966). In 1961, Newman (1979) published *The Structure of Science*, which is a textbook-like presentation of how to do science.

resulted in a restructuration of mathematical theories, which included the formation of new mathematical notions and structures.

Structuralism acknowledges the importance of outlining the alphabet, defining the formulas, and enumerating the axioms for developing a mathematical theory. Structuralism also emphasises the importance of specifying the nature of proof, although without operating with an explicit enumeration of rules of inference. With respect to proving, structuralism sticks to the practice of mathematics, according to which proving must be strictly logical and transparent. There is no application of any form of intuition in mathematical proving; no figures or diagrams are necessary, not even in geometry. In this sense, structuralism assumes the whole approach of eliminating intuition from mathematics.[21]

Nicolas Bourbaki was an important exponent of structuralism. In some places, one can read that he worked at the Royal Academy of Poldavia, in other places that he was associated with the University of Nancago. However, behind the collective pseudonym one finds mathematicians including André Weil (1906–1998), Henri Cartan (1904–2008), Claude Chevalley (1909–1984), and Jean Dieudonné (1906–1992). Over time, many more people have contributed to the collected works of Bourbaki.[22]

The Bourbaki working group was established in the mid-1930s. The original idea was to write a university textbook in mathematical analysis covering recent developments in mathematics. Soon, however, the work became much more ambitious and turned into a project of providing a systematic presentation of major parts of mathematics. The first volume of *Elements of Mathematics* (*Éléments de Mathématique*) was published in 1939 (Bourbaki, 2004). It provides a presentation of set theory, which

21 The formal logical systems, as presented in the *Begriffsschrift* or *Principia Mathematica*, operate with two rules of inference that easily can be stated explicitly. But if we are dealing with a mathematical formal system, such as Peano's axiomatics for the natural numbers, many more rules of inference are going to be applied. But which? One could stipulate that the set of possible inferences for a Peano axiomatics correspond to the theorems in, say, *Principia Mathematica*. This seems consequential, as *Principia Mathematica* presents a system of valid inferences. However, the situation is more complex than that. There are forms of mathematical reasoning which are not captured by any theorem in *Principia Mathematica*, but which are broadly applied in making mathematical deduction. For mathematical research practice, also as shaped by structuralism, the implication is that the rules of inferences are not enumerated, but inferences are kept as transparent as possible.

22 See, for instance, Bourbaki (1950) and Dieudonné (1970).

was considered the basis of mathematics; this idea Bourbaki shares with Frege, Whitehead, and Russell, and logicism in general. Many more volumes of *Elements of Mathematics* followed covering topics like algebra, topology, and topological vector space.

Historically, there is a connection between formalism and Bourbaki's structuralism via Emmy Noether (1882–1935) who, for a period, worked at the mathematical department at Göttingen University, directed by Hilbert. Bartel van der Waerden (1903–1996) was one of her students, and his book *Modern Algebra*, first published in two volumes in 1930 and 1931, was deeply inspired by Noether's lectures. The book is referred to by Dieudonné as an important resource for the Bourbaki group, preparing as it did for the definition of several of the formal structures to which they referred.

The Bourbaki group met a few times per year. At such meetings, manuscripts were presented and discussed carefully, sometimes being read aloud and criticised sentence by sentence. Alternative suggestions for completing a proof were suggested as well as alternative definitions of concepts. The meetings had no chair, and the discussion could be heated. When a manuscript had been worked through, a different member of the group got the task of presenting a revised version of the manuscript at the following meeting. This procedure continued meeting after meeting until consensus was reached.

Only one formal rule guided the work in the group, namely that, on turning fifty years old, the member had to leave the group. New members had to be recruited, and if members became aware of particular gifted students, they could be invited to join a meeting. Any newcomers who did not make significant contributions were dropped, though a second invitation could be considered.

It was presupposed that the members of the group had broad interests in mathematics since the work in the group was not for narrow specialists, the principal aim being to identify relationships between different areas of mathematics. Bourbaki tried to identify how structures and proofs in one area appeared similar to structures and proofs in other areas. When such similarities were identified, the challenge was to make them explicit, and Bourbaki identified a range of such overlapping structures.

Mathematics is in rapid development, new theories and new concepts are constantly emerging. How can we effectively integrate and update all these developments? Bourbaki provided a suggestion. The *Elements of Mathematics* can be read as a kind of mathematical encyclopaedia, organised not in alphabetic order, but structurally.

Chapter 1 in the first volume of *Elements of Mathematics* makes a presentation of what is to be understood by formal mathematics. By making this start, Bourbaki explicitly takes a formalist departure. Chapter 2 presents set theory, defining notions like order pair, function, and correspondence. Chapter 3 addresses ordered sets, cardinals, and integers, while the final Chapter presents the notion of structure.

The crucial notion is *structures*, which became the building blocks of Bourbaki's architecture of mathematics. In order to describe a structure, the properties of its elements are without significance. Bourbaki agreed completely with Hilbert's formulation in the *Foundations of Geometry*, when he enumerated objects like 'point', 'line', and 'plane' without specifying anything about these objects. The only thing relevant is to specify how they relate to each other, and this is done in terms of the axioms defining the structure.

Through Bourbaki's profound studies of a variety of mathematical theories, three *mother structures* were identified: (1) a set organised by an operation; (2) a set organised by a relation; and (3) a set organised by a topology. The group $(G,\circ)$ is an example of a set G organised by an operation $\circ$ which is a function of two variables from $G \times G$ to G. The group $(G,\circ)$ fulfils the axioms:

1. $\forall a,b,c \in G\colon (a \circ b) \circ c = a \circ (b \circ c)$
2. $\exists n \in G\colon \forall a \in G\colon a \circ n = n \circ a = a$
3. $\forall a \in G\colon \exists a^{-1} \in G\colon a \circ a^{-1} = a^{-1} \circ a = n$

One theorem in group theory states that there is only one neutral element. The proof runs like this: Assume that there exists two neutral elements n_1 and n_2. According to the definition, we would have $n_1 = a \circ a^{-1}$ as well as $n_2 = a \circ a^{-1}$. From this we can conclude that $n_1 = n_2$. Group theory developed further along such lines. The departure is the axioms, and nothing but axioms, and the proving needs to be logically straightforward and transparent. The group structures can be recognised in a variety of mathematical disciplines: number theory, geometry, vector calculus, etc.

By means of the mother structures, a huge amount of other mathematical structures can be defined. Notions like ring, field, ordered field, vector space, and Hilbert space can be defined, and the many different classic disciplines start growing together in the same architecture.

By emphasising the importance of mother structures, Bourbaki diverged from a traditional formalist outlook as, for instance, summarised by Haskell Curry (1970) when he states that the 'essence of mathematics lies [...] not in any particular kind of formal system, but in formal structure as such' (p. 56). Bourbaki does not assume any such relativism, but finds that some structures are more important than others to the extent that they express fundamental similarities between apparently different mathematical disciplines. That we are dealing with three mother structures is not any *a prior* given. It is an insight that emerged from the discussions in the Bourbaki group. More mother structures could be identified as mathematics develops. What we are dealing with is just a summing-up of structures identified by a certain group of mathematicians at a certain moment in the history of mathematics.

The Bourbaki group took as its point of departure the current state of mathematics. For identifying structures, they did not consider any historical developments that have brought forward the mathematical ideas and theories. Nor did they pay attention to possible applications of mathematics. Applications were not considered relevant for identifying mathematical structures.

Through a profound de-contextualisation of mathematics, Bourbaki's structuralism repeats the separation between mathematics and socio-political issues as advocated by logical positivism. I interpret structuralism as a principal example of how the dogma of neutrality can be acted out within mathematics research. Most ironically, however, structuralism gained a profound social impact through a widespread reformation of mathematics education.

The Modern Mathematics Movement

The seminar *New Thinking in School Mathematics* took place over twelve days in 1959 at Cercle Culturel de Royaumont, a more than

700-hundred-year-old abbey located north of Paris. The seminar was organised and financed by the Organisation for European Economic Co-operation (OEEC), later to become the Organisation for Economic Co-operation and Development (OECD).

In the peaceful environment provided by the Royaumont Abbey, an important feature of the Cold War was addressed. The tension between the East and the West had been steadily growing, and the military potentials were a crucial factor. The assumption had been, at least in the West, that the USA was well ahead of the Soviet Union with respect to technology in general, and military technology in particular.

One important element of military technology was the capacity for deploying rockets, and it came as a major shock to the West when in 1957 the Soviet Union launched their first Sputnik.

The seminar *New Thinking in School Mathematics* was provoked by the Sputnik shock. It became accepted that in order to advance technology, recognised as an urgent matter, radical improvements in mathematics education were necessary. At the seminar, the mathematician Marshall H. Stone gave the introductory lecture and highlighted that the 'teaching of mathematics is coming to be more and more clearly recognized as the true foundation of the technological society which it is the destiny of our time to create' (p. 18). This and others of his formulations resonated nicely with the overall OEEC rationales for organising the seminar. However, right after the opening lecture, the seminar took an abrupt turn and references to social and technological issues were forgotten.

In his lectures, which turned out to become the principal reference for the whole seminar, Dieudonné presented drastically new ideas about the content of secondary school mathematics. (As mentioned, Dieudonné was born 1906, meaning that he had turned fifty years old and therefore had to leave the Bourbaki group. This might have created space for him to engage in other activities.) He started his lecture this way:

> My specific task today is to examine, from the point of view of present curriculum in mathematics in universities and engineering schools: (a) What mathematical background professors in these institutions would like to find in the students at the end of their secondary school years. (b) What they actually get. (c) How it would be possible to improve the existing situation. (OEEC, 1961, p. 31)

Dieudonné's perspective is clear: a reform of the mathematical curriculum at secondary schools has to be guided by the actual curriculum at the university level. He asks for a radical updating of the curriculum:

> The curriculum of the secondary schools has to be reorganised in order to eliminate any undue waste of time and to absorb as much as possible of the burden now resting entirely of the university as is compatible with the intellectual capacities of the children. (p. 34)

What reorganisation, then?

> In the last 50 years, mathematicians have been led to introduce not only new concepts but a new language, a language which grew empirically from the needs of mathematical research and whose ability to express mathematical statements concisely and precisely has repeatedly been tested and has won universal approval. But until now the introduction of this new terminology has (at least in France) been steadfastly resisted by the secondary schools, which desperately cling to an obsolete and inadequate language. And so when a student enters the university, he will most probably never had heard such common mathematical words as, set, mapping, group, vector space, etc. (p. 34)

Dieudonné wants a conceptual updating of secondary school mathematics. He is not referring explicitly to the work of Bourbaki, but it is clear that his suggestion reflects his structuralist outlook. The curriculum of secondary school mathematics has to be developed around the basic mathematical structures.

This demand, Dieudonné turned into a slogan: 'Euclid must go!' For centuries, Euclid's *Elements* had existed as a principal departure for mathematics education. The *Elements* provided a path whereby proofs led to one theorem after the other. This path has been assumed to reveal the genuine nature of mathematics. But according to Dieudonné, this approach belongs to the museum of mathematics. Euclid must go in order to make space for a relevant updating of the whole discipline.[23]

'Euclid must go!' condenses clearly the structuralist concern with respect to intuition. Intuitions had been incorporated in the whole Euclidean presentation of geometry. This became obvious when Pasch and Hilbert made explicit the many 'hidden axioms' in Euclid's *Elements*. That intuition had brought the deductive processes forward had been

23 See also Dieudonné (1973).

hidden by the presence of diagrams. Diagrams should not have any role to play in mathematics, but in Euclid's *Elements* they did. According to structuralism, this diagram-based intuition had to be eliminated, and in particular structuralist presentations of geometry could be completed without any use of diagrams.

Mathematicians from around the world with an interest in mathematics education joined the seminar. They listened to Dieudonné's presentation, discussed over the twelve days, and gained much inspiration. From Denmark participated Svend Bundgaard, a mathematician from Århus University, and Ole Rindung, particularly interested in secondary school mathematics. At the seminar, it was decided that an expert group should be brought together in order to provide a synopsis for the new curriculum for secondary school mathematics. The group had sixteen members, including Erik Kristensen, also from Århus University. In August–September 1961, the group met in Dubrovnik, and in 1961 their report *Synopsis of Modern Secondary School Mathematics* (OECD, 1961) was published.

A few years later, Rindung and Kristensen published the first volume of a mathematical textbook for the Danish *Gymnasium* for sixteen- to nineteen-year-old students. This textbook was radically different from what had been seen until then. It started with set theory, and right from the beginning the symbolic language of formal logic was brought into operation. The principal mathematical structures were presented, and a new path into the whole landscape of mathematics was defined.

Soon there appeared textbooks for fourteen-year-old students at the Danish *Folkeskole* for six- to sixteen-year-old students, starting with set theory. Simultaneously, textbooks for teacher education and for in-service training of teachers became published, all reflecting the idea that set theory provided the start of learning mathematics. Bent Christiansen from the Royal Danish School of Educational Studies was deeply engaged in implementing the reform by developing material for teachers as well as for students. Soon appeared textbooks for six- to seven-year-old children starting with set theory. In the end, the structuralist approach came to dominate mathematics education in Denmark, at least for a while.

My reference to the development in Denmark serves as an illustration, as what took place in Denmark took place, *mutatis mutandis*, in many other countries as well. We are dealing with a most powerful reform movement. I am not aware of any other educational reforms with such an immediate impact.

The Modern Mathematics Movement covered mathematics education through new structures together with an implicit claim about neutrality, totally distancing it from socio-political issues. Mathematical structures were the focus, not what could be done by means of mathematics. Although the rationale for the Royaumont Seminar was both economic and political, the structuralist outlook annihilated all such 'externalities'. Structuralism focused on intrinsic features of mathematics, and it represented the ultimate de-contextualisation of both mathematics research and of mathematics education. It provided the final step of the ambition of logical positivism of characterising mathematics as neutral, establishing the dogma of neutrality.[24]

Poor Piaget!

In the middle of the 1970s, when the Modern Mathematics Movement was in full swing, and when I started studying mathematics education, one found references to the work by Jean Piaget everywhere. There appeared to exist a clear connection between his formulations of a genetic epistemology and the Modern Mathematics Movement.

When I first looked through the report from the Royaumont Seminar, I was surprised not to find any references to Piaget. It appeared to me that the implementation of the Modern Mathematics Movement came before its epistemological justification. I became interested in clarifying better the nature of Piaget's genetic epistemology. An important resource for me was the book *Mathematical Epistemology and Psychology*, written by Ewert Beth and Jean Piaget (1966), which first appeared in French in 1961. The book is divided into two parts, the first written by Beth, the second by Piaget.[25]

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24 In this way, structuralism cemented the ground-zero from which critical mathematics education was to sprout, to which I return in Chapter 11 of this volume.

25 Beth was deeply interested in the foundation of mathematics. His book of more than 700 pages, *The Foundations of Mathematics* (Beth, 1968), first published in 1959, provides a most elaborated discussion of foundational issues. See also Piaget (1970).

In his part, Piaget refers to a seminar that took place in 1952, in which both Dieudonné and he participated. Dieudonné presented the structuralist view on mathematics as formulated by Bourbaki, and he outlined the nature of the three mother structures. Piaget presented how he had studied children's operations with objects, and how he had condensed his observations by means of three operational structures. Piaget tells that the high degree of correspondence between the three mother structures and the three operational structures appeared as a surprise to those participating in the seminar, and also to Piaget himself.

What can be concluded from such an observation of similarity? One can make a step further than just acknowledging similarities by claiming that there exists an intrinsic connection between the two types of structures. As the mother structures are the basic building blocks in Bourbaki's architecture of mathematics, one can be tempted to stipulate children's operational structures as being the genetic roots of mathematics. To me, this stipulation constitutes the departure for Piaget's genetic epistemology. The seminar in 1952 might be the occasion where this idea emerged.

The idea of a genetic epistemology is original. A classical empirical interpretation of the roots of mathematical knowledge has highlighted that mathematical concepts and insights emerge from *observations of properties* of physical objects. One experiences a very smooth surface, and one gets to the concept of a plain. One makes addition of different objects, and one gets to the basic laws of arithmetic. Piaget's idea is different. He sees *reflections on operations* with objects as being the root of mathematics.

On various occasions, Hans Freudenthal pointed out that Piaget completely misunderstood the nature of Bourbaki's work.[26] According to Freudenthal, Bourbaki's suggestion for an architecture of mathematics just represents a particular event in the history of mathematics. The identified mother structures could have been different; their identification depended on the heated discussions in the Bourbaki group. What ended as the architecture was just a historical coincidence.

To Freudenthal it appears arbitrary, if not simply misunderstood, to conclude that – due to similarities between children's operational

26 In Chapter 7 of this volume, we will look more carefully into Freudenthal's view on mathematics; here I restrict myself to mentioning his critique of Piaget.

structures and some structures identified during the late 1930s by a group of French mathematicians – one had identified the genetic roots of mathematics. To me as well, it appears arbitrary, if not misunderstood. I find that the references to Piaget accompanying the Modern Mathematics Movement first of all served as a questionable legitimisation of what was taking place. Freudenthal (1973) points out the following:

> Poor Piaget! He did not fare much better than Kant, who had barely consecrated Euclidean space as 'a pure intuition' when non-Euclidean geometry was discovered! Piaget is not a mathematician, so he could not know how unreliable mathematical system builders are. Bourbaki's system of mathematics was not yet accomplished when the importance of categories was discovered. There can be little doubt that categories will be a new organizing principle and that rebuilding of Bourbaki's structure in categorical style will leave no stone left on top of another. If a leading development psychologist could then convince us of the categorizing genesis of all mathematical concepts – which will certainly eventually happen – then it will just be in time to see the categorical style mathematics, before it is ready, being pulled down in favour of some new principle, which will certainly have its day. Mathematics is never finished – anyone who worships a certain system of mathematics should take heed of this advice. (pp. 45–46)

Piaget's genetic epistemology recapitulates the complete separation between the learning of mathematics and socio-political issues. His theory is about patterns of 'natural growth', and not about social and critical reflections.[27]

The dogma of neutrality was established as integral part of the formalist outlook on mathematics. From there is became articulated by logical positivism as a much broader dogma of neutrality, not only with respect to mathematics but with respect to science in general. Structuralism represents a further development of the formalist outlook with a profound impact on the mathematical research practice and the formation of mathematical theories. Structuralism embraces the dogma

27 This separation is repeated by Ernst von Glasersfeld's radical constructivism, which represents a further elaboration of Piaget's genetic epistemology. In a conversation with me, Christine Keitel told that once she had the opportunity to ask Glasersfeld how he saw social and political issues related to mathematics education. Glasersfeld found the question interesting, but admitted that he had never thought about it.

of neutrality, and via the Modern Mathematics Movement this dogma became propagated in mathematics education.

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