Studies on Mathematics Education and Society

BREAKING IMAGES

ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

EDITED BY BRIAN GREER, DAVID KOLLOSCHE, AND OLE SKOVSMOSE

https://www.openbookpublishers.com

All external links were active at the time of publication unless otherwise stated and have been archived via the Internet Archive Wayback Machine at https://archive.org/web

Any digital material and resources associated with this volume will be available at https://doi.org/10.11647/OBP.0407#resources

Volume 2 | Studies on Mathematics Education and Society Book Series
ISSN Print: 2755-2616
ISSN Digital: 2755-2624

ISBN Paperback: 978-1-80511-321-8
ISBN Hardback: 978-1-80511-322-5
ISBN Digital (PDF): 978-1-80511-323-2
ISBN Digital eBook (EPUB): 978-1-80511-324-9
ISBN HTML: 978-1-80511-325-6

DOI: 10.11647/OBP.0407

Cover image: *Fall* by Tara Shabnavard
Cover design: Jeevanjot Kaur Nagpal

Published with the support of the Open Access Publishing Fund of the University of Klagenfurt.

7. Intuition revived

Ole Skovsmose

In the preface to Mathematics as an Educational Task, *Hans Freudenthal states that his educational interpretation of mathematics betrays the influence of L. E. J. Brouwer's view on mathematics. In this chapter we explore the nature of this possible influence. According to Brouwer, intuition plays a crucial role in any form of mathematical construction, which he specifies in terms of mental acts. He finds that mathematics does not have any adequate articulation in language, and that mathematical formalisms are nothing but imprecise and mischievous depictions of genuine mathematical processes. Freudenthal characterises mathematics as a* human activity, *thereby subsuming the overall intuitionist outlook that Brouwer had condensed into the notion of* mental activity. *While Brouwer installed intuition in a central position in mathematics, Freudenthal created a vast space for intuition in all kinds of activities in mathematics education. In his writings, Freudenthal does not demonstrate any interest in socio-political issues related to mathematics. Structuralism and the Modern Mathematics Movement are manifestations of the dogma of neutrality, and so is Freudenthal's formulation of mathematics as a human activity. However, although he does not repudiate a dogma of neutrality, he simultaneously provides ideas that help in formulating a critical mathematics education.*

Intuitionism is one prime example of how a conception of mathematics may influence the teaching and learning of mathematics. L. E. J. Brouwer and Hans Freudenthal are two protagonists in this development. Brouwer was a mathematician contributing to a broad range of topics, later focused on formulating an intuitionistic mathematics. For an extended period Freudenthal worked as Brouwer's assistant as a dedicated mathematics researcher, while in the later part of his career he concentrated on mathematics education.

 https://doi.org/10.11647/OBP.0407.07

In the preface to *Mathematics as an Educational Task*, Freudenthal (1973) makes the following comment: 'My educational interpretation of mathematics betrays the influence of L. E. J. Brouwer's view on mathematics (though not on education)' (p. ix).

Let us first look at the side-remark in the parenthesis. How was Brouwer as a teacher? Bartel van der Waerden, who studied mathematics in Amsterdam, makes the following comment about Brouwer:

> I once interrupted him during a lecture to ask a question. Before the next week's lesson, his assistant came to me to say that Brouwer did not want questions put to him in class. He just did not want them, he was always looking at the blackboard, never towards the students. (O'Connor & Robertson, 2003)

Freudenthal's side-remark might not be at all surprising to those who knew Brouwer as a teacher, and therefore let it remain in the parenthesis. What more does Freudenthal tell us in *Mathematics as an Educational Task* about Brouwer's influence? Surprisingly, nothing.[1] In Freudenthal's other books on mathematics education – *Weeding and Sowing* (1978), *Didactical Phenomenology of Mathematical Structures* (1983), and *Revisiting Mathematics Education* (1991) – one finds almost no mention of Brouwer, except for a couple of references. Thus, in Freudenthal's own texts, one does not find a clarification of the nature of Brouwer's influence.[2] Nevertheless, this influence is the focus of this chapter.

Luitzen Egbertus Jan Brouwer (1881–1966) worked in several mathematical areas, including topology, set theory, and measure theory. Brouwer's (1911) contribution to topology includes a theorem that is referred to as Brouwer's fixed-point theorem. It states that for any continuous function f mapping a compact convex set onto itself there exists a point x_0 such that $f(x_0) = x_0$. The theorem is fascinating. When one stirs a cup of coffee – and we assume that the coffee represents a compact convex set, and that the stirring operates like a continuous function – then at least one of the coffee atoms will end up in the same position as it had before the stirring.

In 1912, Brouwer secured a permanent position at the University of Amsterdam, and in his inaugural lecture 'Intuitionism and Formalism'

1 He refers only once more to Brouwer (p. 40), with respect to a different issue.
2 In la Bastide-van Gemert (2015), I did not find any clarification either.

he started articulating more carefully his conception of intuitionism. Brouwer confronted formalism, in the first instance as represented by David Hilbert. This was a confrontation with many ramifications, for instance with respect to the editorial policy of *Mathematische Annalen*, which was the most important international mathematical research journal of the time. From 1902 to 1939, Hilbert was editor, while Brouwer was a member of the editorial board from 1914 to 1928. Due to Hilbert's initiative, Brouwer was removed from the board; other members of the board protested, among them Albert Einstein. The confrontation between intuitionism and formalism was a clash between research paradigms as well as between personalities.

Hans Freudenthal (1905–1990) was born in Germany. In 1923, he started studying mathematics in Berlin, where, in 1927, he met Brouwer, who was giving a lecture. In 1930, Freudenthal completed his doctoral thesis on topology,[3] and soon after he was invited by Brouwer to come to Amsterdam, where during the 1930s he worked as an assistant for Brouwer. After the German invasion in 1940, Freudenthal was suspended from his position due to his Jewish origins. In 1943, he was sent to a concentration camp, but in 1944 through the support of his Dutch wife he managed to escape, and he went into hiding in Amsterdam until the end of the war. After the war, Brouwer was not interested in offering Freudenthal a position again, and in 1946, he took up a position at the University of Utrecht, where he remained for the rest of his career.

Freudenthal was a dedicated mathematics researcher with a specific focus on algebraic topology.[4] However, he did not show any particular dedication to the detailed mathematical elaborations of intuitionistic mathematics. From the late 1960s, Freudenthal started engaging in mathematics education. In 1968, he founded the journal *Educational Studies in Mathematics*, and in 1971 he became nominated as director of the new research institute IOWO, the Dutch abbreviation for *Institut voor de Ontwikkeling van het Wiskunde Onderwijs* (Institute for the Development of Mathematics Education) in Utrecht. By that time, Freudenthal had published widely in mathematics education, and

3 For an important result of this work, see Freudenthal (1931).

4 He proved what are referred to at Freudenthal's spectral theorem and Freudenthal's suspension theorem. Other mathematical conceptions also carry his name.

many of these publications were brought together and reworked into his monumental work *Mathematics as an Educational Task* that appeared in 1973.[5]

In the following, we explore how Brouwer saw *mathematics as a mental activity*. We move on to explore Freudenthal's conception of *mathematics as a human activity* as it came to be expressed in *Mathematics as an Educational Task*. As an indication of what this conception could mean for mathematics education, we look at the example *Ship Ahoy*. As a conclusion we raise the question: What about socio-political issues?

Mathematics as a mental activity

As a way out of the foundational crises in mathematics, Brouwer launched an approach different from those suggested by logicism and formalism.[6] According to him, both logicism and formalism were wrong in their approaches in trying to eliminate intuition from mathematics. The way out of the crisis had to be found in the opposite direction: intuition had to be installed in its proper position as the core of mathematical thinking.

Brouwer found that the emergence of the paradoxes that brought about the foundational crises indicated that something had gone wrong within mathematics itself, and that this problem was manifest in logicism and formalism. What was needed was a much more radical approach. According to Brouwer, the emergence of paradoxes indicates that mathematics has applied forms of reasoning and proof strategies that are not valid in mathematics. Over time mathematics has incorporated a range of theorems, which should not count as such. It is not surprising, then, that paradoxes do appear. The whole body of mathematics had to be re-examined, and for doing so a revitalisation of intuition was needed. This is what Brouwer suggested by formulating an intuitionist conception of mathematics.

5 In 1991, one year after the death of Freudenthal, IOWO was renamed as the Freudenthal Institute. In 2006, due to the integration of more areas, the institute turned into the Freudenthal Institute for Science and Mathematics Education (see van Heuvel-Panhuizen, 2015).

6 For the following presentation of Brouwer's intuitionism, I draw on Ravn and Skovsmose (2019). For a discussion of the foundational crises in mathematics, see Chapter 4 in this volume.

Brouwer saw formalisations as being inaccurate, if not simply misleading. According to him, one can never identify mathematics with any formalism. That would be the same mistake as assuming that a plaster cast of a human being is the actual human being. Mathematics is alive, formalisms are not. Formalisms are only external and imprecise representations of intuitive mental acts, which constitute genuine mathematics.

In 1905, Brouwer (1996) published a short text *Life, Art and Mysticism*, in which he states: 'Always and everywhere truth is in the air, and whenever it breaks through, truth is always the same to those who understand' (p. 404). Brouwer sees truth in absolute terms, and this idea he maintains in his formulation of intuitionistic mathematics. One could think of intuition as being imprecise and open-ended, making space for a variety of interpretations compromising the possible connections between mathematics and certainty. However, Brouwer does not operate with any common-sense interpretation of intuition. He does not relate intuition to uncertainty and ambiguity, but to particular mental acts that bring about mathematical truths with certainty. To him, truth becomes the same to 'those who understand'.

In 1913, Brouwer published his inaugural lecture 'Intuitionism and Formalism'. Here he relates his ideas to those of Immanuel Kant (1973), who in *Critique of Pure Reason*, first published in German in 1781, provided a radical new departure for interpreting mathematics. Kant finds that our experiences become organised according to pre-given categories of understanding, and that mathematics provides the basic structures of the conceptual twins: space and time. That mathematics applies to our experiences of nature is not due to the fact that nature as such operates according to mathematical patterns, but to the fact that mathematics organises our experiences of nature. Brouwer (1913) sees Kant as articulating an intuitionism, but he also highlights that in Kant 'we find an old form of intuitionism, now almost completely abandoned, in which time and space are taken to be forms of conception inherent in human reason' (p. 83).

To Kant, Euclidean geometry reveals details of our category of space. Many interpreted the emergence of non-Euclidean geometries as devastating for Kant's conception of mathematics. Brouwer, however, is not troubled by this critique. He highlights that the position of intuitionism has 'recovered by abandoning Kant's apriority of space but adhering the more resolutely to the apriority of time' (p. 85). For identifying the origin of mathematical intuition, Brouwer put aside any intuition of space, and concentrated on the intuition of time.

From where does an intuition of time emerge? One could think of it in psychological terms. In a *System of Logic*, first published in 1843, John Stuart Mill (1970) argues that all human knowledge, including mathematics, is based on empirical evidence. However, Brouwer does not assume any such psychologism. Like Kant, he sees time as a category for understanding, and not as a psychological notion referring to some particular experiences.

A critical notion to Brouwer is two-oneness. This notion represents the time-specific origin of mathematics. Let us start looking at Brouwer's (1913) own presentation of the notion:

> Neo-intuitionism considers the falling apart of moments of life into qualitatively different parts, to be reunited only while remaining separated by time as the fundamental phenomenon of the human intellect, passing by abstracting from its emotional content into the fundamental phenomenon of mathematical thinking, the intuition of the bare two-oneness. (p. 85)

While Brouwer thinks of Kant's position as an old form of intuitionism, he refers to his own formulation as a neo-intuitionism.[7] He highlights that time is the fundamental phenomenon of the human intellect. Through this formulation he somehow makes space for a psychologism, but immediately distances himself from this position by highlighting that we need to abstract away the emotional content associated with time in order to reach the fundamental phenomenon of mathematical thinking, thus sweeping aside psychological content in order to reach time as a pure category. In this way he gets to the fundamental mathematical phenomenon of mathematical thinking: the bare two-oneness.

7 Brouwer acknowledges that there are several sources of inspiration for this new form of intuitionism, and, with reference to controversies with respect to the interpretation of mathematical laws, he refers to '*intuitionism* (largely French) and *formalism* (largely German)' (Brouwer, 1913, p. 82). Brouwer also makes references to Henri Poincaré and Émile Borel, who together with Henri Lebesgue and several others have been referred to as semi-intuitionists (see Troelstra, 2011).

In 'Intuitionism and Formalism' Brouwer does not give any further explanation of why he uses the expression 'two-oneness', and not, say, 'one-twoness'. It would seem that the latter expression would indicate more directly the start of the counting process. However, there might be linguistic reasons for Brouwer's choice of terminology. He might be alluding to the notion of 'trinity'. In Dutch the word for trinity is *drie-eenheid*, which literally means 'three-oneness'. Later, as for instance in the *Cambridge Lectures*, Brouwer talks about a 'twoity', where the allusion to trinity is even more explicit.

The intuition of movement of time in terms of two-oneness is the basic departure for mathematical thinking. Any mathematical concept becomes created by this intuition:

> This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise still further to the smallest infinite ordinal number ω. (pp. 85–86)

Brouwer uses the formulation 'the basic intuition of mathematics, creates...'. The notion 'creates' is crucial, it is a mental process that constructs mathematical entities staring out from the intuition of two-oneness. The two-oneness is not an intuition through which one discovers mathematical truths. It is an intuition through which one constructs mathematical entities and mathematical truths. One can think of the two-oneness as referring to the first step in a process of counting: one, two. This process can be repeated, and one counts: one, two, three. It can be repeated again and again: one, two, three, etc. Brouwer does not accept the concept of actual infinity, but assumes the idea of potential infinity. The sequence of natural numbers can be indefinitely extended. It is in this sense we need to read Brouwer's claim that the counting process gives rise to 'the smallest infinite ordinal number ω'.

What about geometry? Brouwer has put aside intuition of space as being an irrelevant category, as he finds that also geometric notions are also developed from the intuition of time:

> The apriority of time does not only qualify the properties of arithmetic as synthetic a priori judgments, but it does the same for those of geometry,

> and not only for elementary two- and three-dimensional geometry, but for non-Euclidean and n-dimensional geometries as well. (p. 86)

Kant's principal point is that mathematical statements are synthetic *a priori* judgements. Brouwer shares this idea with respect to time. That mathematical statements have a content and simultaneously are independent of empirical observations, is due to the fact that the intuition of time ensures an *a priori* structuring and simultaneously provides mathematical statements with a synthetic content.

After outlining the basic ideas of intuitionism, Brouwer continues in 'Intuitionism and Formalism' to address the paradoxes that provoked the foundational crises in mathematics. He points out that within an intuitionistic approach such paradoxes will evaporate. For instance, the intuitionistic restrictions with respect to the construction of sets will imply that the set-theoretical paradox that was identified by Bertrand Russell and Ernst Zermelo will disappear.[8] Thus Brouwer tries to demonstrate that intuitionism establishes a solid route out of the foundational crises.

After the presentation of 'Intuitionism and Formalism', Brouwer elaborated intensively on all aspects of the intuitionist program, and he gave series of lectures. In 1926, he lectured in Göttingen, which was the most prominent place for mathematical research, directed by Brouwer's principal opponent, Hilbert. In 1927, he lectured in Berlin, where Freudenthal was in the audience. In 1928, he lectured in Vienna, where Ludwig Wittgenstein was attending and got inspired to return to philosophy. In 1934, Brouwer lectured in Geneva, and during the years 1947–1951, he gave a series of lectures in Cambridge. His intention was to organise these lectures in a book, and he completed five of the planned six chapters. They became published posthumously as *Brouwer's Cambridge Lectures on Intuitionism* (Brouwer, 1981).

In 'Intuitionism and Formalism' (1913), he gave an opening outline of intuitionism, while the *Cambridge Lectures* can be read as his more reflected formulations. Here Brouwer (1981) uses the terminology that mathematics develops through particular acts. In this way, he highlighted explicitly the constructivist nature of intuitionism. He presents what he refers to as the *first act of intuitionism* in the following way:

8 See Chapter 4 in this volume, for a presentation of this paradox.

> Intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics. (pp. 4–5)

As in 'Intuitionism and Formalism', Brouwer refers to a 'falling apart of a life moment' as constituting the origin of mathematics. He talks about a twoity that when stripped of particular emotional qualities, turns into an 'empty form of the common substratum of all twoities'. We are dealing with a pure twoity, which represents the basic intuition of mathematics. It signifies the first mental act of intuitionism. It is the same intuition that Brouwer previously had referred to as a two-oneness.

Brouwer claims that intuitionistic mathematics is essentially a languageless activity. However, intuitionistic mathematics also becomes expressed through symbols, and I assume that Brouwer did write something at the blackboard when giving his Cambridge lectures. But still, according to intuitionism, this is just chalky shadows of what mathematics really is: *a languageless activity of the mind*.

In the *Cambridge Lectures*, Brouwer presents a *second act of intuitionism*, which is also a way of creating new mathematical entities:

> In the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it, definitions of equality having to satisfy the conditions of symmetry, reflexivity and transitivity. (p. 8)

This second act refers to ways of creating species of already created mathematical entities. Brouwer does not use the notion of set, but one can think of species as a collection of entities being 'equal' to each other.

Brouwer claims that all mathematics can be constructed through the two acts of intuitionism; no other pattern of construction is necessary. This is the clue to Brouwer's constructive interpretation of mathematics.

Many traditional forms of mathematical inferences are not valid from an intuitionist point of view. Mathematics has been all too tolerant by applying inferences which are not guided by the acts of intuitionism. Let us consider a classic proof of the theorem T: *There exist infinitely many prime numbers*. The negation $\neg T$ states: *There exists a maximum prime number* that we can refer to as P. Let us assume $\neg T$. Let the sequence of prime numbers smaller than P be $p_1, p_2, \ldots, p_n$. We define a new number N as $N = p_1 \times p_2 \times \ldots \times p_n \times P + 1$. As for any number, N can be uniquely factorised as the product of prime numbers. Consider one of these prime numbers, which we can call Q. Q cannot be any of the numbers $p_1, p_2, \ldots, p_n, P$, as a prime number cannot be a factor in two consecutive numbers. It follows that Q must be bigger than P. By assuming $\neg T$, we reach a contradiction. As a consequence, we conclude T: There exist infinitely many prime numbers.

Brouwer does not accept indirect proving, as this does not represent a constructive way of binging about a mathematical entity or a mathematical truth. Assuming a Platonist position, the set of natural numbers is a pre-existing entity, and so is the set of prime numbers. Either the set of prime numbers is finite, or it is infinite. Only these two alternatives are possible. If one assumes that there exists a maximum prime, and this leads to a contradiction, the alternative must be true. But this is not a constructive proof, according to Brouwer. If one wants to prove T, then one has to provide a construction that leads to T. One could easily be in a situation where one cannot prove T or $\neg T$, and according to Brouwer, neither T nor $\neg T$ is true until one of them has been proved constructively.[9]

According to intuitionism, then, the whole body of existing mathematical theories needs a careful revision, which includes three elements. First, one needs to consider what classic mathematical results can be considered valid within an intuitionistic mathematics.

9 Brouwer (1981) makes the following observation: 'The belief in the universal validity of the principle of the excluded third in mathematics is considered by the intuitionists as a phenomenon of the history of civilisation of the same kind as the former belief in the rationality of π, or in the rotation of the firmament about the Earth' (p. 7). The validity of is nothing but a cultural phenomenon that can be explained along the same lines as many other superstitions. There is nothing in this logical formula except long-lasting preconceptions. However, Brouwer does acknowledge that in some particular domains the principle of the excluded middle does work: it could be in everyday situations; it could also be in some more particular mathematical cases. But as a general principle to be used in mathematics, it is illegitimate.

Second, one needs to consider which classic proofs can be reformulated and given new constructive formats. Third, one has to consider what parts of classic mathematics cannot be transferred into intuitionistic mathematics. Georg Cantor's (1874) theory of sets, which leads to the idea of an infinity of infinities, is an obvious candidate. Through such a re-examination, mathematics will be cleansed of invalid results, and possible paradoxes will be eliminated.

Let us consider again the classic proof of the existence of infinitely many prime numbers. It applies the principle of excluded middle, and is therefore not constructive. But the theorem can be reformulated and the proof reorganised to meet constructivist standards. The theorem can be stated as: *For any prime number P, it is always possible to construct a prime number Q that is bigger than P*. Define *N* as in the non-constructivist proof above and let *Q* be a prime factor of *N*. It follows that *Q* must be bigger than *P*. This formation is in accordance with intuitionism, not assuming any actual infinity. Through the very proving, we have constructed the prime number *Q* bigger than *P*, and we can conclude: For any prime number *P*, it is always possible to construct a prime number *Q* bigger than *P*.[10]

In 1975 and 1976, Brouwer's collected works appeared in two volumes. The first, *Collected Works, Vol 1: Philosophy and Foundations of Mathematics* is edited by Arend Heyting.[11] The second, *Collected Works, Vol. 2: Geometry, Analysis, Topology and Mechanic*s is edited by Freudenthal. The two editors, Heyting and Freudenthal, are real insiders of intuitionistic mathematics.[12]

10 The reformulation of the classic proof for the infinity of prime numbers was not a big deal, as the classic proof already contained the constructive features; it just had to be reformulated. However, there are mathematical theorems that are much trickier. For instance, what about Brouwer's own fixed-point theorem? He made the proof according to classic standards; however, Kellogg, Li and Yorke (1976) 'saved' the theorem by giving a constructive proof. Brouwer's fixed-point theorem makes part of intuitionistic mathematics.

11 Heyting (1971) provides a captivating introduction to intuitionism.

12 Intuitionistic mathematics has had a tremendous development. Mathematical analyses have been developed according to an intuitionistic outlook (Bishop, 1967; Lorenzen, 1971; Martin-Löf ,1968). It has turned out that this approach has a particular significance for computing (Martin-Löf, 1982). Intuitionism has paved the way for a new richness of philosophic discussions (Dummett, 1977; Lorenzen 1969). Intuitionistic logic, as formalised by Heyting (1930) to the great consternation of Brouwer, got related to other logical structures by Gödel (1933), and came to play a crucial role as a logic relevant for computer science (see Reeves

Summary of Brouwer's conception of mathematics

No mathematical entity or mathematical truth exists before it has been constructed. This claim opposes the ontology of any form of Platonism, which assumes that mathematical entities have a real existence, independent of human intervention. To Brouwer, processes of obtaining mathematical knowledge are processes of construction, not processes of discovery.

According to Brouwer, intuition plays a crucial role in any form of mathematical construction. This intuition he specifies in terms of two mental acts. Brouwer does not think of such acts as taking place in a specific mind. He does not present mental acts in psychological terms, and does not suggest any form of what could be referred to as psychological constructivism.[13] Nor does Brouwer's constructivism includes any trace of social constructivism. The mental acts Brouwer has in mind do not presuppose any interaction; they are idealised individual acts; and they bring about the same entities and the same truths for 'those who understand'.

An intuition of time is a fundamental phenomenon in human life, and after abstracting away all emotional features of the movement of time, we reach the fundamental phenomenon of mathematical acting: the intuition of the naked two-oneness, also referred to as a twoity. While the first act of intuitionism takes the form of counting, the second act takes the form of groupings of already constructed mathematical entities. According to Brouwer, all mathematics can be constructed through these two acts.

Mathematics is languageless. Mathematics does not have any adequate articulation in language, and mathematical formalisms are nothing but imprecise and at times mischievous depictions of genuine mathematical processes. Mathematical processes are alive, while mathematical formalisms are dead and distorted copies. Mathematics is a languageless activity of the mind.

---

and Clarke, 2003). For a general overview of the development of intuitionism, see Troelstra and Dalen (1988).

13 Compared to Brouwer's constructivism, Jean Piaget's constructivism is psychological by highlighting the importance of the mental processes of assimilation and accommodation for the construction of knowledge.

Brouwer's conception of mathematics means a revitalisation of intuition as a crucial feature of mathematics. By doing so, Brouwer confronts formalism, which tried to eliminate intuition from mathematics. Formalism saw intuition as the cause of the foundational crisis in mathematics, Brouwer sees intuition in terms of well-defined mental acts, as saving mathematics from contradictions.

Mathematics as a human activity

Brouwer's ideas did not directly bring changes to mathematics education. However, some of his ideas became re-elaborated by Freudenthal, who opened a new terrain for activities in mathematics education by making plenty of space in which for intuition to operate. Before Freudenthal, other Dutch mathematics educators sought inspiration in intuitionistic ideas, and such visions for mathematics education had been presented in the Dutch mathematics teacher education journal *Euclides*.[14] However, these visions faded away, while Freudenthal's elaboration turned out to have a profound impact.

We are going to consider Freudenthal's conception of mathematics as expressed in *Mathematics as an Educational Task*.[15] Freudenthal sees mathematics as a human activity, while Brouwer sees it as mental acts.[16] We will point out similarities and differences between these two conceptions.[17] We will try to clarify what Freudenthal referred to when, in the Preface, he mentioned that his educational interpretation of mathematics betrays the influence of Brouwer's view of mathematics.

14 Let me refer to two publications: Rootselaar (1957) and Heyting (1957). Heyting observes that intuitionism might have an educational relevance, as several intuitionistic concepts come close to students' natural perceptions. Both papers focus on intuitionism as a source of inspiration for mathematics teachers, not as a proper goal in mathematics education. I do not read Dutch, but Danny Beckers has provided me with these references and a short summary of them.

15 Other important contributions by Freudenthal that we also could address are Freudenthal (1978, 1983, 1991).

16 See Gravemeijer and Terwel (2000) for a careful presentation of what Freudenthal means by mathematics being a human activity.

17 We have to be aware of a principal difference in the presentation of the two conceptions. While Brouwer presents his conception explicitly, as in 'Intuitionism and Formalism' and in the *Cambridge Lectures*, Freudenthal's main focus in *Mathematics as an Educational Task* is to formulate a view on mathematics education, rather than present an explicit conception of mathematics.

When Brouwer launched his view, formalism was in powerful development, establishing itself not only as a philosophy of mathematics, but also as an emerging mathematical research paradigm. Brouwer's intuitionism was up against this powerful opponent identifying formal structures with mathematics itself. Freudenthal was also up against formalism, specifically in the form of structuralism as advocated by Bourbaki and acted out through the Modern Mathematics Movement. Freudenthal did not see formal structures as providing a proper departure for mathematics education; instead, students should be involved in mathematical activities.

Freudenthal refers to the Socratic method, which highlights the importance of developing understanding through the students' own activities. He formulates this idea in the following way:

> I will suppose as Socrates did that the teaching matter is re-invention or re-discovery in the course of teaching. Rather than being dogmatically presented, the subject matter originated before the students' eyes. (p. 101)

Freudenthal's critique of a delivery-education can be compared to Paulo Freire's (1972) critique of banking education. Freire criticises profoundly the idea that education means bringing parcels of assumed knowledge to the students, and Freudenthal expresses a similar critique.

The Socratic method is presented in Plato's dialogue *Menon*, in which Socrates talks with Menon's slave.[18] The point of the dialogue is that Socrates does not teach the slave anything. Socrates only puts questions, so no 'transfer' of knowledge is taking place. Starting from these questions, the slave reaches a mathematical insight. This dialogue illustrates Plato's idea that learning means remembering. We can interpret the example as embedded in a Platonic outlook, according to which any kind of obtaining mathematical knowledge takes the form of discovering some truths about an already exiting mathematical reality. This means that any form of mathematical learning becomes a re-discovery, or a dis-covery.[19]

18 See *The Internet Classics Archive*, http://classics.mit.edu/Plato/meno.html

19 Kollosche (2017) provides a detailed analysis of the notion of discovery and dis-covery addressing the Platonic features that might be included in these notions.

While Brouwer would certainly oppose any such interpretation of learning mathematics, Freudenthal is not explicit in formulating an anti-Platonic position. However, I am tempted to interpret Freudenthal's reference to the Socratic method not as an assumption of any Platonism, but more as a general interpretation of learning as being resourced by interaction, communication, and dialogue. In making such an interpretation, Freudenthal certainly distances himself from Brouwer. One can think of Freudenthal as assuming a social interpretation of constructivism, contrary to Brouwer's individual constructivism. I see Freudenthal's reference to the Socratic method in this light. However, we also have to be aware that Freudenthal does not refer to his own interpretation of learning mathematics in terms of constructivism. This is a label, however, that I feel tempted to apply.

Being constructivist does not imply being relativist. To Brouwer, mental constructions of mathematics will lead to the same mathematics. There is only one form of intuitionistic mathematics. Brouwer has inserted an absolutism into his anti-Platonic constructivism. It might be possible to find shades of the same absolutism in Freudenthal's conception of mathematics. This absolutism appears when Freudenthal presents learning as a guided activity, which leads to an insight in already established mathematical knowledge. Freudenthal not only uses the notion of re-discovery, but also the notion of re-invention. By talking about re-invention and not just about invention, Freudenthal makes clear that he does not think of learning mathematics as a process that brings about new mathematical insight in any objective interpretation, but in a subjective. This process brings about new mathematical insight for the students.

Freudenthal does not use the notion of construction, but other related notions – such as activity, creative inventions, direct invention, and re-invention – that bring the message:

> Today, I believe, most people would agree that no teaching matter should be imposed upon the students as a ready-made product. Most present-day educators look on teaching as initiation into certain activities. Science at its summit has always been creative inventions, and today it is even so at levels lower than that of masters. The learning process has to include phases of direct invention, that is, of invention not in the objective but in the subjective sense, seen from the perspective of the students. (p. 118)

Freudenthal talks about a ready-made product being imposed on students, and with such a remark he points his finger at the Modern Mathematics Movement. Through this movement, the whole curriculum became predefined through the structural architecture of mathematics. Again and again, Freudenthal criticises this approach. I suspect he is being ironic when he states that today it is broadly agreed that 'no teaching matter should be imposed upon the students as a ready-made project'. When *Mathematics as an Educational Task* was published in 1973, the Modern Mathematics Movement was still in full swing, although difficulties in its implementation had become recognised.

When Freudenthal describes processes of learning mathematics, he uses several expressions referring, not to the final and polished mathematical structures, but to the processes that can lead to mathematical understanding. Freudenthal changes the focus from 'what to teach' to 'how to learn'. He highlights that the 'learning process has to include phases of direct invention'. Invention, however, is not to be understood in absolute terms, but always with reference to the students' horizons. Freudenthal finds it crucial that students experience that mathematical insight becomes developed from within, and not imposed on them.[20]

Brouwer also concentrates on mathematical processes and refers to mental acts. However, Freudenthal has a much broader conception of mathematical activity in mind. I have no doubt that he was fully aware of the very specific interpretation of mathematical construction provided by Brouwer, and that he did not want to assume Brouwer's metaphysics with respect to the nature of mental acts. To Brouwer the mathematics-creating mental acts are individual; no trace of social interaction can be located in these acts. Freudenthal's conception of mathematics as human activity is different. Formulating arguments, addressing possibilities, evaluating results are all features of mathematical activities, seen as

20 Gravemeijer and Terwell (2000) make this point clearly in the following way: 'As a research mathematician, doing mathematics was more important to Freudenthal than mathematics as a ready-made product. In his view, the same should hold true for mathematics education: mathematics education was a process of doing mathematics that led to a result, mathematics-as-a-product. In traditional mathematics education, the result of the mathematical activities of *others* was taken as a starting point for instruction, and Freudenthal (1973) characterised this as an *anti-didactical inversion*. Things were upside down if one started by teaching the result of an activity rather than by teaching the activity itself' (p. 780).

social processes among students and teachers. This is pointed out by Freudenthal through his reference to the Socratic method. Freudenthal sees the role of the mathematics teacher, not as being a lecturer, rather as being a supervisor helping the students to come to participate in mathematical activities.

By highlighting that we are dealing with a human activity, Freudenthal also stresses that mathematics is not an activity presupposing some particular abilities. It is a common activity. Everybody can participate in a mathematical activity. Freudenthal provides the conception of activity with a broad inclusivity, while Brouwer's mental acts appear exclusive, reserved for 'those who understand'.

Freudenthal talks about 'connected mathematics', and 'lived-through realities', which is very different form talking about mathematical structures:

> To teach connected mathematics it is not wise to start out looking for direct connections; they should rather be found between the contact points where mathematics is attached to the lived-through reality of the learner. Reality is the framework to which mathematics attaches itself, and though these are initially seemingly unrelated elements of mathematics, in due process of maturation connections will develop. Let the mathematicians enjoy the freewheeling system of mathematics – for the non-mathematicians the relations with the lived-through reality are incomparably more momentous. (p. 77)

Freudenthal's clue is that it is not wise to start out looking for direct connections. The point of departure is not any mathematical structures already elaborated by others, but the students' lived-through realities that include mathematics fraught with relations.

> When speaking about mathematics fraught with relations, I stressed the relations with a lived-through reality rather that with a dead mock reality that has been invented with the only purpose of serving as an example of application. This is what often happens even in arithmetic teaching. I do not repudiate play realities. At a low level games may be useful means of motivation. But it is dangerous to rely too much on games. Ephemeral games are no substitutes for lived-through reality. The rules of games that are not daily exercised are easily forgotten as mathematics or even faster. The lived-through reality should be the backbone which joins mathematical experiences together. (pp. 78–79)

By referring to a 'dead mock reality', Freudenthal not only criticises the Modern Mathematics Movement, but also the school mathematics tradition.[21] In this tradition exercises invented by textbook authors play a particular role: Peter has to buy 4.5 kilos of apples … A family is driving on holiday with the average speed of 70 km per hour … The shadow of the flag post is 4.6 meter long … All such exercises are pure inventions; they do not represent any lived-through realities, rather stereotypical didactical inventions.

Brouwer is a radical anti-Platonist. The existence of any mathematical entity or mathematical truth has to be constructed. Before being constructed, nothing exists. This claim brought him to abandon classic logic. Apparently, Freudenthal shares Brouwer's disregard for formal logic. But while Brouwer is very specific in his critique of formal logic, Freudenthal simply makes space for all kinds of reasoning as forming part of mathematical activities.

Summary of Freudenthal's conception of mathematics

If mathematics is an activity, it is not defined by any Platonic reality, nor by any logical or formal structures. Instead of activity, one can also try to use the notion of construction and think of mathematics as a human construction. I find that Freudenthal operates with a constructivist perspective on mathematics, although he does not use this label.

Brouwer did not include any relativism in his version of constructivism, nor does Freudenthal seem to. While the construction of mathematics through research might represent objective inventions, the construction established through education represents subjective inventions.

Whereas Brouwer confronted formalism as represented by David Hilbert, Freudenthal confronted structuralism as represented by Bourbaki and the Modern Mathematics Movement. Confronting formalism and structuralism means giving value to intuition, and both Brouwer and Freudenthal do so. While Brouwer installed intuition in

21 For a characterisation of the school mathematics tradition, see Skovsmose and Penteado (2016).

a central position in mathematics, Freudenthal made a vast space for intuition in all kinds of educational activities.

Brouwer did not see formal logic as capturing the nature of mathematical reasoning. Freudenthal shared this idea, however in *Mathematics as an Educational Task* I do not see traces of Brouwer's way of arguing for this position. Freudenthal acknowledges the different patterns of mathematical reasoning, but he never shows interest in trying to capture a universal pattern of this reasoning. Freudenthal is rather interested in exploring a broad spectrum of intuitive mathematical reasoning in educational contexts.

Many times, Freudenthal characterises mathematics as a human activity. By talking about human *activity*, he assumes the overall intuitionist outlook that Brouwer had condensed in the notion for mental *activity*. By talking about *human* activity and not about *mental* activity, Freudenthal also distances himself from Brouwer. While mental activity refers to highly idealised constructive processes, human activity refers to real-life interactive processes of creating mathematical understanding.

Freudenthal's conception of mathematics means a revitalisation of intuition in mathematics education. It might be this revitalisation that Freudenthal had in mind when in the preface to *Mathematics as an Educational Task*, he mentioned that his educational interpretation of mathematics betrays the influence of Brouwer's view on mathematics.

Ship Ahoy

The Modern Mathematics Movement was guided by a well-defined conception of mathematics: mathematics is formed by its structures, and three basis structures, also referred to as mother structures, had been identified by the Bourbaki group. According to Jean Piaget, three similar structures characterise children's operations with objects, which brought him to assume that he had identified the genetic routs of mathematics. This assumption provided the whole Modern Mathematics Movement with an outstanding legitimisation: the structural organisation of mathematics shows also the natural way of learning mathematics. Freudenthal considered this justification to be nonsense.[22]

Seeing mathematics as human activity opposes directly the conception of mathematics that guided the Modern Mathematics Movement. As an illustration of what this could mean, I refer to an example published in *Five Years IOWO*, published as a special issue of *Educational Studies* in 1976 when Freudenthal retired (Freudenthal et al., 1976).

Ship Ahoy is for children around ten to eleven years old. The whole project is planned to last for about ten lessons. *Ship Ahoy* starts with the children listening to a communication between two ships, Bermuda (B), a yacht, and Constance (C), a tug. The storm makes it sometimes difficult to hear what is said:

> C: Do not read you. Repeat. Over.
> B: This is Bermuda. This is Bermuda. We are in danger, in danger. The motor has failed ... (noise) ... Cast the anchor, but the chain can break any moment. Over.
> C: I read you. What is your position? Over.
> B: Do not know, do not know. Wemelringe area. Probably Wemelringe area. No vision. Over.
> C: Do you see the coast? Over?
> B: Yes, we ... (noise) ...
> C: I do not read you. I do not read you. Over.
> B: We see a lighthouse in the distance, lighthouse in the distance. Over.
> C: We read you. Do you see a church tower? A church tower? Over.
> B: Only water. Only water. Over.
> C: Keep looking and call in. Over.
> B: Yes. A church tower to the left of the lighthouse! Over.
> C: Good, we have your approximate position. We are on our way. On our way. Over.
> B: Thank you. Please hurry. Over.
> C: We are on our way. Keep looking. There is a small house to the right of the lighthouse. Keep looking. Over and out.

In 1977, when I first time read the presentation of *Ship Ahoy*, I was surprised: Could this be mathematics? I am sure that I was not the only one being surprised. At that time, the perception of mathematics was dominated by the Modern Mathematics Moment, which operated with

22 See Chapter 4 in this volume, for a short presentation of Piaget's position and of Freudenthal's critique of Piaget.

a clear idea of what counted as mathematics. This idea was shaken by this and other examples presented by IOWO. Freudenthal's conception of mathematics, as formulated in *Mathematics as an Educational Task*, become both concrete and provocative.

The work in the classroom begins: What is the situation? What could happen? Why is Bermuda in difficulties? What can they see from Bermuda? The children are presented with some pictures showing the lighthouse, the church, and the small house in different positions. Could any of these pictures show the situation as observed from Bermuda? A map of the area is handed out. It shows the position of the lighthouse, the church, and the small house. The map has to be read and properly understood, and then comes the question: Where might Bermuda be located?

Could readings of maps and spatial reasoning be considered mathematical tasks? In 1976, this was hardly considered mathematics. In Denmark, a short textbook for students around fifteen years old had been published, giving a strict axiomatic presentation of incidence geometry. Here lines were defined as sets of points and illustrated as sets conventionally are, within egg-shaped circles. Two non-overlapping eggs illustrated two parallel lines, and so on. The deduction from the presented axioms observed strict formalities. No intuition with respect to points and lines were necessary; such intuitions were in fact considered disturbing for the deduction. An initial part of incidence geometry was carefully elaborated, and the majority of students were completely lost. Compared to such an approach to geometry, looking at maps and speculating about possible perspectives expand the scope of mathematical activities enormously. From being marginalised, intuition moves to the centre of mathematical reasoning.

The intuition cultivated in *Ship Ahoy* concerns three-dimensional space and three-dimensional geometry. The general assumption, associated with traditional mathematics education as well as with the Modern Mathematics Moment, was that one needed to start with two-dimensional geometry and only later get to three-dimensional geometry. When paying particular attention to intuition and not to any axiomatic organisation of geometry, this order turns artificial. We live in a three-dimensional space. All our daily-life experiences are located in such a space. So why not start out with issues related to our three-dimensional space of life? That is precisely what *Ship Ahoy* does.

The rescue work continues. Bermuda is found, and Constance takes her on tow. However, it has become night before they reach the harbour. How to keep the right course? From Constance, one can see the two lights in the Harbour. How the two lights are placed in the harbour can be seen on a map of the harbour also handed out to the children. One light is positioned higher up than the other. How should the captain on Constance see the positions of the two lights in order to keep the right course? The children become engaged in such discussions, and the rescue work continues. Freudenthal talked about starting from situations fraught with relations, and *Ship Ahoy* is an illustration of what this could mean.

The inspiration from Freudenthal and IOWO spread world-wide. By the late 1970s, the inspiration had reached Denmark, where the Modern Mathematics Movement had been broadly implemented. The Freudenthal and IOWO approach showed alternatives, and intuition got revitalised in mathematics education.[23]

What about socio-political issues?

In 1967, I graduated from a teacher education college in Denmark, where I had been carefully introduced to the Modern Mathematics Movement. In 1968, I started studying mathematics at university, and here I encountered a structuralist approach where, for instance, the introductory course in mathematical analysis began with abstract topology.

In 1977, I was accepted as a PhD student at the Royal Danish School of Educational Studies, which concentrated on in-service training of teachers. Since the beginning of the 1960s, the Modern Mathematics Movement had been broadly introduced in Denmark, not least due to the dedicated work of Bent Christiansen from that institution. However, Christiansen became much inspired by Freudenthal's work, and he directed a major change in mathematics education in Denmark.

23 The notion of *realistic mathematics* has been coined and elaborated in detail at the Freudenthal Institute. See, for instance Gravemeijer (1994), De Lange (1987), and Streefland (1991).

Christiansen and Tage Werner were my supervisors, and Christiansen told me about Freudenthal and about IOWO, and he showed me a copy of *Five Years IOWO*. During his whole career, Werner had been a consistent anti-formalist, providing a range of suggestions for engaging students in mathematical activities. He was in line with IOWO even before *Five Years IOWO* was published.

The aim of my PhD project was to formulate a critical mathematics education, and soon after I got started my supervisors made it possible for me to visit IOWO in Utrecht and to meet with Freudenthal. I was anxious. At that time my English was not very good, Freudenthal was so famous, and I was overawed.

Freudenthal met me with a welcoming smile, and I felt relaxed. His enthusiasm was evident when he shared various possible mathematical activities. When I tried to explain about my project and wanted to ask how he viewed the connection between socio-political issues and mathematics education, he seemed, however, uninterested. I did not insist, so our conversation remained focused on possible mathematical activities. Through this interaction, I experienced the richness of educational ideas that emerge from viewing mathematics as a human activity.

I would have liked to insist on my question. Freudenthal uses the notion of *lived-through reality*, which I find to be powerful. It can be given a range of interpretations. The reality for whom? One could think of a *lived-through socio-political reality*. Such a conception can be related to Paulo Freire's notion of *generative themes*, which opens towards a huge variety of mathematical activities with political significance. While Freudenthal talks about mathematics as a *human activity*, one could consider what it could mean to talk about mathematics as a *political activity*.

The notion of lived-through reality can be related to critical mathematics education, but in my meeting with Freudenthal, he was not interested in addressing any such possibility. Nor do I locate any interest in his writings. Structuralism and the Modern Mathematics Movement are manifestations of the dogma of neutrality. They operate *as if* mathematics is neutral and mathematics education can be kept separate from socio-political issues. Freudenthal operates with the same *as if*. He formulates mathematics as an educational task within an apolitical outlook. However, although he embraces a dogma of neutrality, he simultaneously provides notions and ideas that help in formulating a critical mathematics education.

Acknowledgements

I want to thank Danny Beckers for detailed comments and suggestions. He has also drawn my attention to some publications in Dutch, a language that I do not read.

References

la Bastide-van Gemert, S. (2015). *All positive action stars with criticism: Hans Freudenthal and the didactics of Mathematics.* Springer.

Bishop, E. (1967). *Foundations of constructive analysis.* McGraw-Hill.

Brouwer, L. E. J. (1911). Über Abbildungen von Mannigfaltigkeiten [About mappings of manifolds]. *Mathematische Annalen, 71*, 97–115. https://doi.org/10.1007/BF01456931

Brouwer, L. E. J. (1913). Intuitionism and formalism. *Bulletin of the American Mathematical Society, 20*(2), 81–96. https://doi.org/10.1090/S0002-9904-1913-02440-6

Brouwer, L. E. J. (1981). *Brouwer's Cambridge lectures on intuitionism* (D. van Dalen, Ed.). Cambridge University Press.

Brouwer, L. E. J. (1996). Life, art, and mysticism. *Notre Dame Journal of Formal Logic, 37*(3), 389–429. https://doi.org/10.1305/ndjfl/1039886518

Cantor, G. (1874). Über eine Eigenschaft des Inbegriffes aller reelen algebraischen Zahlen [About a property of the epitome of all real algebraic numbers]. *Journal für die reine und angewandte Mathematik, 77*, 258–262. https://doi.org/10.1515/crll.1874.77.258

Dummett, M. (1977). *Elements of intuitionism.* Clarendon Press.

Freire, P. (1972). *Pedagogy of the oppressed.* Penguin.

Freudenthal, H. (1931). Über die Enden topologischer Räume und Gruppen. *Mathematische Zeitschrift, 33*, 692–713. https://doi.org/10.1007/BF01174375

Freudenthal, H. (1973). *Mathematics as an educational task.* Reidel.

Freudenthal, H. (1978). *Weeding and sowing.* Reidel.

Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures.* Reidel.

Freudenthal, H. (1991). *Revisiting mathematics education: China lectures.* Kluwer.

Freudenthal, H., Janssen, G. M., Sweers, W. J., van Barneveld, G. B., Bosman, J. N., van den Brink, F. J., van Bruggen, J. C., Gilissen, L. J. M., Goffree, F., de Gooijer-Quint, J. F., ter Heege, J., Jansen, H. M. M., de Jong, R. A., Kremers, W. H. M., Leenders, C. P., Meijer, G. H., de Moor, E. W. A., Schoemaker, G., Streefland, L., … Wijdeveld, E. J. (1976). Five Years IOWO [Special issue]. *Educational Studies in Mathematics, 7*(3).

Gödel, K. (1933). Eine Interpretation des intuitionistischen Aussagenkalküls [An interpretation of the intuitionistic propositional calculus]. *Ergebnisse eines mathematischen Kolloquiums, 4*(39–40), 300–301.

Gravemeijer, K. (1994). *Developing realistic mathematics education.* CD-ß Press, Freudenthal Institute.

Gravemeijer, K. J., & Terwel, J. (2000). Hans Freudenthal: A mathematician on didactics and curriculum theory. *Journal of Curriculum Studies, 32*(6), 777–796. https://doi.org/10.1080/00220270050167170

Heuvel-Panhuizen, M. van den. (2015). Freudenthal's work continues. In C. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 309–331). Springer. https://doi.org/10.1007/978-3-319-17187-6_18

Heyting, A. (1930). Die formalen Regeln der intuitionistischen Logik [The formal rules of intuitionist logic]. *Sitzungsberichte der Preussischen Akademie der Wissenschaften*, 42–56, 57–71, 158–169.

Heyting, A. (1957). Intuïtionisme en schoolwiskunde [Intuitionism and school mathematics]. *Euclides, 33*(1), 1–12.

Heyting, A. (1971). *Intuitionism: An introduction.* North Holland.

Kant, I. (1973). *Critique of pure reason.* MacMillan.

Kellogg, R. B., Li, T. Y., & Yorke, J. A. (1976). A constructive proof of the Brouwer fixed-point theorem and computational results. *SIAM Journal on Numerical Analysis, 13*(4), 473–483.

Kollosche, D. (2017). Entdeckendes Lernen: Eine Problematisierung [Discovery learning: A problematisation]. *Journal für Mathematik-Didaktik, 38*(2), 209–237. https://doi.org/10.1007/s13138-017-0116-x

Lange, J. de. (1987). *Mathematics, insight and meaning.* Utrecht University.

Lorenzen, P. (1969). *Normative logic and ethics.* Bibliographisches Institute.

Lorenzen, P. (1971). *Differential and integral: A constructive introduction to classical analysis.* University of Texas Press.

Martin-Löf, P. (1968). *Notes on constructive analysis.* Almquist & Wiksell.

Martin-Löf, P. (1982). Constructive mathematics and computer programming. In L. J. Cohen, J. Los, H. Pfeiffer, & K.-P. Podewski (Eds.), *Logic, methodology and philosophy of sciences VI.* North-Holland.

Mill, J. S. (1970). A system of logic. Longman.

O'Connor, J. J., & Robertson, E. F. (2003) *Luitzen Egbertus Jan Brouwer.* https://mathshistory.st-andrews.ac.uk/Biographies/Brouwer

Ravn, O., & Skovsmose, O. (2019). *Connecting humans to equations: A reinterpretation of the philosophy of mathematics.* Springer. https://doi.org/10.1007/978-3-030-01337-0

Reeves, S., & Clarke, M. (2003). *Logic for computer science.* Department of Computer Science, Queen May and Westfield College, University of London, and Department of Computer Science, University of Waikato, New Zealand. http://www.cs.ru.nl/~herman/onderwijs/soflp2013/reeves-clarke-lcs.pdf

Rootselaar, B. van. (1957). Intuïtionisme en rekenkunde [Intuitionism and arithmetic]. *Euclides, 32*(6), 193–198.

Skovsmose, O., & Penteado, M. G. (2016). Mathematics education and democracy: An open landscape of tensions, uncertainties, and challenges. In L. D. English & D. Kirshner (Eds.), *Handbook of international research in mathematics education* (pp. 359–373). Routledge. https://doi.org/10.4324/9780203448946-22

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research.* Kluwer. https://doi.org/10.1007/978-94-011-3168-1

Troelstra, A. S. (2011). History of constructivism in the 20th century. In J. Kennedy & R. Kossak (Eds.), *Set theory, arithmetic, and foundations of mathematics: Theorems, philosophies* (pp. 150–179). Cambridge University Press. https://doi.org/10.1017/CBO9780511910616.009

Troelstra, A. S., & Dalen, D. van. (1988). *Constructivism in mathematics: An introduction.* North-Holland.