Studies on Mathematics Education and Society

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## ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

EDITED BY BRIAN GREER, DAVID KOLLOSCHE, AND OLE SKOVSMOSE

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# 9. The case of Ramanujan: Investigating social and sociomathematical norms outside the mathematics classroom

*Felix Lensing*

*Ever since mathematics education research has 'divorced' from the discipline of mathematics and set out to become a discipline in its own right, there has been a constant debate about what can and should be understood by mathematics education research. In this chapter, I start from the assumption that mathematics education research necessarily takes a 'reflexive stance' towards its objects of study: mathematics education research is not simply engaged with mathematics, but rather with the engagement with mathematics. It investigates the complex interplay of bodily, cognitive, and social processes that are involved in the genesis of mathematical knowledge – especially (but by no means only) when this genesis occurs in educational contexts. Against this background, I will examine the particular role that the distinction between social and sociomathematical norms may play in the empirical study of the social aspects of this genesis. To do so, I will proceed in two steps: I will first detach the distinction between social and sociomathematical norms from its 'conceptual tie' to mathematics classroom practice. Then, I will use the famous correspondence between mathematicians Srinivasa Ramanujan and G. H. Hardy as an example to show how the distinction may offer a fresh perspective on mathematical practices outside the mathematics classroom.*

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## Introduction

In the second half of the last century, mathematics education research emancipated itself from the discipline of mathematics and set out to become a research discipline in its own right. An important insight that paved the way for this emancipation was the recognition that it is not mathematics itself, but rather the doing of mathematics, encompassing mathematical activity in all its different forms and contexts, that constitutes the field of study of mathematics education research. The mathematics education researcher does not simply see what those who participate in mathematical practices see, he or she does not focus on mathematical objects and their manifold relations, but rather examines the underlying 'processes of objectification' (Radford, 2013), that is, the processes in which these very objects and relations are constituted in the first place. As a consequence, mathematics education research does not produce mathematical knowledge, but knowledge about the production of mathematical knowledge. It does not, for instance, formulate and substantiate knowledge claims about mathematical objects, but it seeks to better understand the bodily, cognitive, and social conditions of these formulations and substantiations. It could perhaps be said that mathematics education research facilitates a reflection of mathematical practice upon itself. And it is, of course, particularly interested in mathematical activities as they take place in educational contexts. Once one adopts this 'reflexive stance' and no longer focuses only on the mathematics but rather on the bodily, cognitive, and social processes that underlie it, a whole new field of inquiry opens up. Now all sorts of extra-mathematical factors come into view that regulate these processes and thus also influence what 'comes out' as mathematics in the end.

In this chapter, I want to show how the distinction between social and sociomathematical norms (Voigt, 1995; Yackel & Cobb, 1996), a conceptual tool originally designed for analysing mathematical classroom practice, can be used to examine some of these extra-mathematical factors. Taking the social practice of mathematical research as an example, I will attempt to show that said distinction is also appropriate for the analysis of mathematical practices outside the educational context. Such an analysis, however, requires a generalisation of the distinction between social and sociomathematical norms. So, before it can be applied to all kinds of mathematical practices, the distinction must first be detached from its 'conceptual tie' to the mathematics classroom. In order to achieve this, I will begin with some theoretical considerations concerning the question of what can be understood by norms in general (Section 2). Then, I will introduce the distinction between social and sociomathematical norms as a further subdivision in the realm of norms, thus removing the restriction of the distinction to mathematical classroom practice (Section 3). Finally, I will take the famous correspondence between mathematicians Srinivasa Ramanujan and G. H. Hardy from the beginning of the last century (Berndt & Rankin, 1997) as an example to show how the distinction between social and sociomathematical norms may offer a fresh perspective on mathematical research practice (Section 4).[1]

## On the concept of norm

In the attempt at pinpointing the concept of norm, one will inevitably be faced with the problem that norms appear in the most diverse forms. There are cultural, legal, political, educational, linguistic, industrial, and moral norms, to name just a few. But what is the pattern that connects? What, for instance, do linguistic norms have in common with industrial norms? And what do these two share with moral norms? A common answer to these questions is: Whether linguistic, industrial, or moral, all these norms determine the way in which certain other things should exist. Norms do not say how things *are*, but how they *ought to be*. Linguistic norms dictate how signs of a language *ought to be* used, industrial norms prescribe how products *ought to be* manufactured, and moral norms determine how we *ought to be* acting. It is quite tempting to simply define norms by the factual presence of this peculiar 'ought character': Whenever one comes across something that determines how something else *ought to be*, let's call it a norm. Such definition is of course

1 Note that the aim of this chapter is not to reconstruct the story of Srinivasa Ramanujan's life, but to learn something general about the practice of mathematical research from the individual case of Ramanujan. What aspects of the *person* Ramanujan are relevant to my analysis, and thus what constitutes the *case* of Ramanujan (in the sense intended here) will become clear over the course of this chapter.

possible and also frequently being used.[2] But it leaves the social genesis as well as the social function of norms unexplored. It leaves unexplored how and under which circumstances (e.g., in which social relations of power and control) norms acquire their peculiar 'ought character' and what is gained thereby.

Niklas Luhmann, hence, has proposed to define the concept of norm in a different way (Luhmann, 1995, pp. 319–325). He begins with a more general concept – that of mutual expectation – and then asks: What is the essential quality that is added when a mutual expectation becomes a norm? In what way is it altered by its normalisation? His surprising answer is: not at all. Whether or not a mutual expectation is a norm, cannot be decided by any analysis – however detailed – of its qualities. Rather, it depends exclusively on how the mutual expectation is treated in the case of its disappointment: while mutual expectations of *cognitive character* are abandoned or, at least, altered in case of their disappointment, *normative* ones are being retained even when disappointed (see Luhmann, 1995, pp. 320–321). The normativity of a norm lies in its counterfactual stabilisation: whether or not the world events correspond to it, the norm is left unchanged. Normative expectations have a sort of 'built-in safeguard' that prevents them from being modified. It can thus be anticipated what to do in case of their disappointment, namely: hold on to them. From this analysis it follows that the peculiar 'ought character' of norms is merely a consequence of a more fundamental property, that of counterfactual stabilisation. Norms specify how something ought to be because factual violations have no consequences on them, i.e., do not lead to their alteration.

What this analysis has not yet addressed is the question of what norms are for: What is their social function? Which social problem is solved by protecting mutual expectations against their alteration? Luhmann's answer to those questions is: through the technique of normalisation, even highly uncertain expectations are able to obtain social validity. If one appeals to norms, then one can assure in the here and now 'that one will not be left helpless by disappointment or reveal oneself as someone who simply does not know the world and harbored false expectations'

2 For example, Hans Kelsen (1959) writes: 'Now, what is a norm? A norm is a specific meaning, the meaning that something ought be, or ought to be done, although actually it may not' (p. 107).

(Luhmann, 1995, p. 320). Instead, the trajectories for how things might continue are already clearly mapped out.

An example may illustrate that: a common norm in mathematics classrooms is that students ought to pay close attention to class. But teaching experience shows over and over again that this mutual expectation is being disappointed. Despite this obvious uncertainty of expectation, though, normalisation allows teachers and students to be prepared for these events of disappointment: Teachers, for instance, may think of disciplinary measures and, in addition, can be sure in advance to be able to justify having taken those measures. Likewise, students who are being held responsible for a classroom interruption can assume that it will be sufficient to indicate their readiness to reinstate the very norm they have just disappointed. All that is required is an apology after the fact and accepting imposed measures to rehabilitate the violated norm. Even denying a norm violation – or at any rate, its personal attribution ('Gee, but it wasn't me, Mrs. Baker') – ultimately only confirms the violated norm and thus serves to reinforce it.

## On the distinction between social and sociomathematical norms

Now that I have discussed some of the aspects that characterise norms in general, I want to introduce a further subdivision into the realm of norms, namely the distinction between *social* and *sociomathematical norms*. Whether in family life, educational contexts, or mathematical science, whenever a *mathematical practice* arises two types of norms can be delineated within the norms that govern the behaviour occurring in that practice: 1) those norms that regulate the behaviour with reference to its mathematical content, and 2) those norms that regulate it without such reference. While I will refer to the first type of norms as *sociomathematical norms*, I will call the second type *social norms*. To take up the above example: the norm that students ought to pay close attention to class is a *social norm* because it regulates the classroom practice without any reference to its mathematical content. It defines a general boundary between legitimate and illegitimate behaviour in the classroom, albeit a kind of 'generalised' one which is valid not only within the mathematics classrooms but across all school subjects. In contrast, the question of what counts as a mathematical argument in a particular classroom, for instance, refers to a *sociomathematical norm*. This question can only be answered by recourse to the mathematical content as it is thematised in classroom communication.

The distinction between social and sociomathematical norms was originally introduced as a conceptual tool to investigate norms in school mathematics classrooms (Voigt, 1995; Yackel & Cobb, 1996).[3] Later, it was also used to seek for normative orders in mathematics education contexts at university level (Yackel et al., 2000). With my determination of the distinction above, however, I am aiming at giving up its 'conceptual tie' to the educational context altogether. Naturally, mathematics instruction at all different educational levels remains a potential field of application for the distinction; but I am convinced that, in principle, any mathematical practice can be examined for its social and sociomathematical norms. In the remainder of this article, I will support this conviction with an exemplary analysis of some social and sociomathematical norms of mathematical research.

Before I turn to this exemplary analysis it is, however, necessary to highlight an important methodological implication from the preceding theoretical considerations: if norms can be characterised by their counterfactual stabilisation, then situations in which they are violated are of particular interest in reconstructing norms. This point was also highlighted by Anna Sfard:

> A norm becomes explicit and most visible when violated. Violation evokes interlocutors' spontaneous attempts at correction, often accompanied by a condemnation of the transgressor's illegitimate behavior. (Sfard, 2010, p. 204)

In short, anyone who wants to investigate the social and sociomathematical norms of a particular mathematical practice should be looking for situations in which a norm violation occurs.

3 To be more precise: For Cobb, Yackel, and colleagues, the distinction played more than a purely analytical role; from the outset, it was linked to questions of how to develop new forms of mathematics instruction. As a consequence, they were, for example, also concerned with the question of how to give social validity to certain norms to which they wanted to orient the instruction in their project classrooms. In this chapter, however, I am primarily concerned with the distinction as an analytical tool for the empirical study of mathematical practice.

## The case of Ramanujan

An extreme case of norm violation occurred at the beginning of the last century. In January 1913, Godfrey Harold Hardy, at the time a professor at the University of Cambridge and one of the leading mathematicians in the fields of calculus and number theory, received a letter from a young Indian mathematician named Srinivasa Ramanujan. The letter began with the following words:

> Dear Sir,
>
> I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. [...] I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as ›startling‹. [...] I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. [...]
>
> I remain, Dear Sir, Yours truly,
>
> S. Ramanujan (Berndt & Rankin, 1997, pp. 21–22)

Ramanujan was a mathematical genius without any direct exposure to the specialised culture of European mathematics (‘I have had no University education’). The ‘enclosed papers’ consisted of nine densely written pages on which Ramanujan presented a selection of his mathematical findings. He had arrived at his – as it later turned out, groundbreaking – findings mainly through self-study.[4] Only a few mathematics books served him as a base (Berndt & Rankin, 2000). Ramanujan’s explicitly formulated goal was to publish his mathematical findings, probably also to earn some money (‘Being poor, if you are convinced that there is anything of value I would like to have my theorems published’). Ramanujan’s position as ‘mathematical outsider’, that is to say, his role as someone who had barely experienced guided forms of mathematical enculturation, makes this case an ideal object

4 Accounts of his life can be found in Rao (1998) and Kanigel (1992).

of study.[5] Because who could commit greater norm violations than someone who has encountered prevailing norms only implicitly, namely through the study of a few selected books?

Hardy replied to Ramanujan's letter the following month:

> I was exceedingly interested by your letter and by the theorems which you state. You will however understand that, before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions.
>
> Your results roughly seem to fall into three classes:
>
> 1. there are a number of results which are already known, or are easily deducible from known theorems;
> 2. there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance;
> 3. there are results which appear to be new and important, but in which almost everything depends on the precise rigour of the methods of proof which you have used. (Berndt & Rankin, 1997, p. 46)

The short excerpt of Hardy's letter shows that in the presentation of his findings Ramanujan had violated several norms at once: some of his findings were not new or, at least, easily derivable from known theorems. His mathematical results were missing proofs. And Ramanujan himself did not seem to know which of his findings were merely interesting, and which were of great mathematical importance.

I am now going to consider these three aspects with regard to the social and sociomathematical norms that regulate the acceptance of mathematical findings for publication: I will deal with the norm of mathematical novelty first (a), then turn to the question of what counts as a valid result in mathematics (b), and finally deal with the question of how a particular finding can obtain mathematical importance or significance (c).

## On the novelty of mathematical findings

5 Note that it is this one 'abstract' aspect of the *person* Ramanujan (his being a 'mathematical outsider' in the sense described above) that is relevant to the analysis conducted here, namely, to the reconstruction of some of the social and sociomathematical norms being valid in mathematical research at the turn of the century.

A major requirement for mathematical findings in order to be accepted for publication is *mathematical novelty*. Findings that are published in mathematical journals should neither be already known nor should they be direct consequences from what is already known ('there are a number of results which are already known, or are easily deducible from known theorems'). Novelty practically means to disappoint expectations. For something new to emerge, one must deviate from the paths already walked in the epistemic processes. Without such deviation from the expected, no new mathematical knowledge can evolve. By elevating mathematical novelty into a necessary condition for publication, mathematics embraces the unexpected. In a sense, it forces itself to learn. Mathematics cannot reject mathematical findings because it does not know anything about them yet. If new mathematical findings arise, then mathematics is compelled to expand its knowledge. The boundary between the known and the unknown is redrawn with every mathematical publication. In this successive advancement of knowledge, not only mathematical knowledge increases, but also what is yet mathematically unknown. Each newly developed mathematical theory leads to further mathematical problems. Every solved mathematical problem generates a multitude of resultant problems. As David Hilbert (1902) once put it: 'It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon' (p. 438).

Since any decision concerning the novelty of a mathematical finding can only be made on the basis of the current state of mathematical knowledge, mathematical novelty is a sociomathematical norm. However, this sociomathematical norm is based on a social norm: it holds true for science in general that it forces itself to shift the boundary between the known and the unknown in consideration of new findings. That mathematics aims at surprising itself is thus a norm that it shares with other sciences.

## On the validity of mathematical findings

However, their mathematical novelty is not enough for mathematical findings to be published. Another important question is how to decide upon the validity of mathematical results.[6] To answer this question, first hints can be drawn from the excerpt of Hardy's letter. Hardy points out that he can only make a final judgment about the scientific value of Ramanujan's mathematical work with access to the proofs for his findings ('You will however understand that before I can judge properly of the value of what you have done, it is essential that I should see proofs of some of your assertions').[7] It becomes evident from his choice of words that Hardy is not referring to a mere personal expectation here, but to a norm generally valid in mathematics. By prefacing his argument with the phrase 'You will however understand that', it becomes clear that Ramanujan is expected to be able to rehabilitate the norm he violated. The said norm can perhaps be formulated as follows:

> (1) Whether a mathematical finding is valid or not ought to be decided on the basis of mathematical proof.

First, it must be emphasised that this norm is an evolutionary achievement of mathematics: by no means has it always been the case that the validation of a mathematical finding had to be carried out on the basis of proof.[8] Moreover, a comparison with other scientific disciplines

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6 This question, of course, is at the heart of the traditional understanding of philosophy of mathematics, or more precisely: epistemology of mathematics, and one could fill entire libraries with books written on the question of the justification of mathematical knowledge. Hence, in the following I will limit myself to only those few aspects that appear in the correspondence, and I ask the reader's indulgence for falling far short of the level of discussion reached in the philosophy of mathematics. Indeed, my goal in this chapter is not so much to contribute to this discussion, but simply to show that the distinction between social and sociomathematical norms can be used as a conceptual tool in the empirical study of mathematical research practice.

7 Comparing Hardy's response with Ramanujan's letter, it is noticeable that the term 'value' has undergone a subtle semantic transformation: while in Ramanujan's letter 'value' also seems to be linked to an economic aspect ('Being poor, if you are convinced that there is anything of value I would like to have my theorems published'), this aspect no longer appears in Hardy's answer (see above). Here, the term 'value' seems to be used solely in the sense of 'scientific value'.

8 For example, one reads in Kleiner (1991) about Babylonian mathematics: 'Babylonian mathematics is the most advanced and sophisticated of pre-Greek mathematics, but it lacks the concept of proof. There are no general statements in Babylonian mathematics and there is no attempt at deduction, or even at reasonable explanation, of the validity of the results. This mathematics deals with specific problems, and the solutions are prescriptive – do this and that and you will get the answer' (p. 292). For a more comprehensive account of Babylonian mathematics, see also the work of Jens Høyrup, particularly Høyrup (2002).

shows that this norm is a sociomathematical norm. It is true for science in general that findings always require a scientific justification. Just like in the case of mathematical novelty, the sociomathematical norm refers to a more general social norm. But there are very few disciplines besides mathematics in which proof plays such a prominent role in the context of justification. If one asks for the validity of a mathematical finding, one is thus referred to a mathematical proof. But if it is the proof upon which the validity of the mathematical finding rests, then the question arises as to when a mathematical proof can be considered as valid. The sociomathematical norm as expressed in (1) may thus be specified as follows:

> (2) Whether a mathematical finding is valid or not ought to be decided by examining the validity of the associated mathematical proof.[9]

But what are the criteria for a mathematical proof to be valid? The answer can only be: it depends. Even a brief look at the history of mathematics leads to the conclusion that the criteria for validity of a proof have changed again and again in the socio-cultural evolution of mathematical research (Calude et al., 2003; Chemla, 2015; Kleiner, 1991; MacKenzie, 1999). History of mathematics is rife with 'incomplete' proofs that were 'completed' by mathematicians of a following generation, only to be exposed as incomplete again and so on.[10] But if leading mathematicians of any generation repeatedly come up with incomplete proofs, this can only be a sign that the underlying validity criteria change over time.

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9 This could be a starting point for historical studies that reconstruct the transformation of the validity criteria for mathematical proofs over the course of time. For the purposes of this analysis, however, such a purely formal description of the sociomathematical norm shall suffice.

10 As an example may serve the proof history of the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra states that every non-constant single-variable polynomial with complex coefficients has at least one complex root. With regard to its proof history, Kline (1990) says: 'Proofs offered by d'Alembert and Euler were incomplete. In 1772, Lagrange, in a long and detailed argument, 'completed' Euler's proof. But Lagrange, like Euler and his contemporaries, applied freely the ordinary properties of numbers to what were supposedly the roots without establishing that the roots must at worst be complex numbers. Since the nature of the roots was unknown, the proof was actually incomplete. The first substantial proof of the fundamental theorem, though not rigorous by modern standards, was given by Gauss in his doctoral thesis of 1799 at Helmstädt' (p. 598).

> The notion of proof is not absolute. Mathematicians' views of what constitutes an acceptable proof have evolved. [...] The validity of a proof is a reflection of the overall mathematical climate at any given time. (Kleiner, 1991, p. 291)

Based on the previous considerations, this metaphorical description of validity criteria as 'a reflection of the overall mathematical climate at a given time' can be further clarified: validity criteria are consolidated as sociomathematical norms on the ground of the ongoing acceptance and rejection of mathematical findings in mathematical publication practice. The normative requirements that must be met in the presentation of mathematical findings are not imposed on mathematics from the outside. Rather, they arise from within. It is mathematics itself that writes the norms to the sky that guide the publication process. These norms do not have an absolute character, but their normative character means precisely that they can only be changed in the longer term. And only because these expectations are always already found as valid norms by every mathematician who wants to participate in mathematical communication, mathematicians can also use them as a sort of self-control device. Hardy, for instance, can expect himself to have certain expectations about the presentation of mathematical findings by other mathematicians only because the relevant norms have already acquired validity in the social practice of mathematics. Regular participation in this social practice (e.g., the reading, writing, and reviewing of mathematical papers) leads to socialisation effects in the minds of participants. In this way, mathematicians learn what is expected from them when they present their mathematical results to other mathematicians and vice versa.

## On the importance of mathematical findings

The acceptance of a mathematical finding for publication is thus conditioned by at least two normative aspects: first, the finding must be a mathematical novelty, and second, it must be accompanied by a valid mathematical proof. There is, however, a third aspect mentioned in the passage of Hardy's letter that influences the publication process: the *mathematical importance* of a finding. In his classification of Ramanujan's new mathematical results, Hardy distinguishes between merely interesting and important results ('there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance'). There are countless new and interesting truths in mathematics, but only a few of them are also mathematically important. But what exactly does this mean? What is the mathematical importance of a finding? 'We may say, roughly, that a mathematical idea is ›significant‹ if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas' (Hardy & Snow, 2004, §11). Hardy thus argues that the yardstick for the mathematical importance of a mathematical finding is therefore its 'mathematical connectivity'. The greater the number of mathematically important ideas to which a certain finding can be connected, the greater its mathematical importance. Whether a particular finding is mathematically important or not, thus, can often only be decided in retrospect. It depends on how and to what extent further mathematically important results can be connected to it. While the novelty or validity of a mathematical finding can already be judged with a certain degree of certainty in the here and now, many times its importance becomes apparent only in the future. Mathematical results will have been important. They often acquire their importance only from a certain point in the future, from which it becomes clear that they were the basis for a multitude of further mathematically important results. This inscribed reference to the future prevents the value of mathematical importance from becoming a necessary decision criterion in the publication process. Concrete examples are the works of Évariste Galois on the theory of polynomial equations, Hermann Grassmann's grounding of what was later called linear algebra, or Gottlob Frege's founding of modern logic, all of which had in common that they were hardly noticed, let alone appreciated, by their contemporaries.

With such an analysis, however, the relation between mathematics and time is still insufficiently grasped. Mathematics constantly projects findings from the present into the future. Mathematical relationships are permanently explored on the basis of hypotheses. If a given hypothesis were true, then this set of propositions could be derived from it. This way, in many cases one can already know in the present that the proof of a certain mathematical hypothesis in the future would be of greatest mathematical importance. Whoever solves one of the so-called millennium problems, for example, is guaranteed mathematical fame. This ‘anticipated’ importance of a mathematical finding has an influence on the publication process. Although mathematical importance cannot be elevated to a necessary condition for publication due to its immanent reference to the future, it can, at least, influence where (i.e., in which mathematical journals) a particular result is published. In contemporary mathematics it makes a considerable difference whether a finding is published in the *Annals of Mathematics* or in the *Mathematische Annalen*. Mathematics introduces a rank order within its field of acceptance: it establishes a hierarchy of mathematical findings through the distribution of publications among the various journals.

## Conclusion

I have set out to show that the distinction between social and sociomathematical norms can shed light on some of the extra-mathematical factors involved in the production of mathematical knowledge. For this purpose, I took a series of steps: I first criticised the common practice of defining norms by their peculiar ‘ought character’ and argued that norms are better understood as a specific kind of mutual expectation. Norms differ from all other kinds of mutual expectations in that they are retained in cases of disappointment. This characterisation does not simply replace the ‘ought character’ of norms but explains it. If a mutual expectation is counterfactually stabilised, i.e., if factual violations do not lead to a norm’s modification, then one is able to know in advance (and independently of what is actually done) what is ought to be done. Based on these theoretical considerations about norms in general, I then introduced the distinction between social and sociomathematical norms: while sociomathematical norms are those norms of a mathematical practice that regulate the participants’ behaviour with reference to some mathematical content, social norms do so without such reference. Compared to the original conception of this distinction, which limited its scope to educational contexts, this characterisation is an attempt at extending the distinction to all kinds of mathematical practices. This extension was based on the following assumption: while there may be significant differences between different mathematical practices in terms of *which* social and sociomathematical norms are established, *that* such norms emerge is a universal feature that is common to all mathematical practices. To support this assumption, I then analysed the correspondence between Ramanujan and Hardy and showed that social and sociomathematical norms are 'active' not only in mathematics classrooms but also in mathematical research practice. By analysing the role that sociomathematical norms of mathematical novelty and validity play in the evaluation of mathematical findings, I showed that social and sociomathematical norms are often intertwined with each other without, however, coinciding. It is, for example, a social norm common to all scientific disciplines *that* findings are in need of justification, but *how* they are to be justified is governed by sociomathematical norms specific to mathematical research practice. This peculiar relationship was also noted in the context of mathematics education. Erna Yackel and colleagues (2000), for instance, provide the following examples:

> The expectation *that* one is to give an explanation falls within an analysis of social norms, but *what* is taken as constituting an acceptable mathematical explanation falls within an analysis of sociomathematical norms. Likewise, the expectation *that* one is to offer a solution only if it is different from those already offered falls within the realm of social norms, but *what* is taken as constituting mathematical difference falls within the realm of sociomathematical norms. (p. 282)

So we always have, on the one hand, a social norm that says *that* something should be done and, on the other hand, a correlated sociomathematical norm that tells us *how* it should be done. In all of these cases, sociomathematical norms specify social norms for the particular context, and, conversely, sociomathematical norms are 'backed up', so to speak, by more general social norms. But since there are also social norms, such as the norm that students should follow class attentively, that can stand for themselves, that do not require any further specification by a sociomathematical norm, the question arises: under what conditions does this special relationship between social and sociomathematical norms occur? The empirical analyses have led us to the conceptual limits of the distinction; they revealed that further distinctions are needed to account for all facets of the normative orders of mathematical practices.

Moreover, the analysis of the correspondence between Ramanujan and Hardy has also led to the conclusion that by no means all social structures of mathematical research practice are norms. The example of the value of mathematical importance made it quite clear that the distinction between social and sociomathematical norms captures only a small 'section' of the extra-mathematical factors involved in the production of mathematical knowledge. It is thus an important question for further research addressing the distinction between social and sociomathematical norms to focus on the relation of these two types of norms to other kinds of social structures (e.g., to what Sfard, 2010, pp. 200–208, calls 'metadiscursive rules' or what Voigt, 1985, 1995, calls 'patterns of interaction').

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