Studies on Mathematics Education and Society

BREAKING IMAGES

ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

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11. A critical conception of mathematics

Ole Skovsmose

A critical conception of mathematics emerged through several routes: this chapter takes a closer look at three of them. First, we follow how the students' movement, beginning in the late 1960s, inspired a critique of university studies in mathematics. This analysis turned into a critique of mathematical modelling, emphasising that it is an illusion that mathematics ensures objectivity and neutrality. It became recognised that, when brought into action, mathematics may have all kinds of technological, economic, and political impacts, including many of the most questionable kind. Second, we see how mathematics becomes recognised as a plurality of constructions. I show that mathematics is shaped through social, historical, cultural, and political – in short, human – processes, and that any uniform conception of mathematics is a deception, if not a falsification. Third, I illustrate how mathematics can be developed as a critical resource and become a means for identifying forms of economic and political oppression. Mathematics can play a part in the struggle for social justice.

Since Antiquity, mathematics has been admired and celebrated, while a critical conception of mathematics has become clearly formulated only within the last century.

Plato put mathematics on a pedestal, as to him it revealed what it could mean to enter the world of ideas. Euclid's *Elements* brought together an axiomatisation of geometry that, right up to the late nineteenth century, was celebrated as showing a perfect exposition of mathematics as well as a pattern for presenting knowledge in general. In Europe the admiration for mathematics gained additional fuel through the so-called scientific revolution. The people contributing to this – Nicolas Copernicus, Galileo Galilei, Johannes Kepler, and Isaac Newton, to name a few – were all

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deep believers in God. They saw the world as created by God, implying that an insight into and understanding of nature are an insight into and understanding of God's creation. Apparently, this creation had been completed according to mathematical principles. After the natural sciences separated from religious beliefs, the celebration of mathematics continued, with mathematics becoming nominated as the language of science.

In this chapter, we leave behind any such unexamined admiration of mathematics and look for the emergence of a critical conception of mathematics. Elsewhere in this volume, different conceptions of mathematics have been outlined, for example those of Godfrey Hardy, Nicolas Bourbaki, George Pólya, Ludwig Wittgenstein, L. E. J. Brouwer, and Hans Freudenthal. However, when it comes to the formulation of a critical conception of mathematics, it is not possible to provide any such name-related simplification.

In many common-sense interpretations, critique means pointing out flaws, weaknesses, and problematic issues. This forms part of a critical activity, but a critique can also point out strengths, advantages, and positive qualities. This is generally assumed when one talks about film critique, literary critique, art critique, and this also applies when one talks about a critical conception of mathematics. We will follow three routes towards a critical conception of mathematics: seeing mathematics as a multiplicity of actions, as a plurality of constructions, and as a critical resource.[1]

Mathematics as a multiplicity of actions

The extreme optimism that accompanied modernity assigned a crucial role to science and technology as being the true motors of the progress that would ensure economic welfare, a richer cultural life, and a

1 More routes lead to the formulation of a critical conception of mathematics, including those addressed by Houman Harouni in the next chapter. Feminism represents another such route, see Chapter 19 in this volume and also Leone Burton (1995) on a feminist epistemology of mathematics, and Gabriele Kaiser and Pat Rogers (1995) for providing a broader overview of the movement. Also, critical race theory (see Chapter 19) and related movements open for a critical conception of mathematics, as addressed in the work of Danny Martin and his colleagues (Martin, 2013, 2019; Martin & Gholson, 2012).

permanent state of peace. This optimism, that reached its peak by the turn of the nineteenth century was, however, not for everybody. It was based on ignoring the atrocities caused by colonialism and the brutal exploitations of workers. Such reservations being disregarded, the world was seen as enjoying steady progress. Science and technology could be celebrated, and the general admiration of mathematics seemed well-grounded.

The outbreak of the First World War called into question and undermined this narrow-minded optimism. In the most dramatic way, this war demonstrated that science and technology form an integral part of the machinery of war. Airplanes were constructed for military purposes, and chemical weapons used for the first time in history. It is true that for centuries mathematicians had worked on ballistics, but now it became greatly more obvious how mathematics contributes to the further development of war technologies.[2]

It would seem that the time had come where the blind admiration of mathematics could, and should, be questioned. But it was not. It took some more time before a proper critical conception of mathematics became formulated.

Fachkritik

Emerging during the late 1960s, the students' movement advanced a broad spectrum of political ideas, also about university education. Demands emerged for a new organisation of this education, not according to traditional disciplines. It should address social issues; it should be problem-oriented; it should be project-organised. The students should have a principal say in what to study, and how to study it. Professorial dominance should be broken, and the topics taught at the university had to be subjected to a profound critique.

In German this critique was referred to as a *Fachkritik*; in Danish we have a similar word *fagkritik*, but I have never seen any adequate English translation. *Fach* means school subject or discipline, and *Fachkritik* refers to a critique of subject matter issues. One can be specific and talk about

2 For discussion of relationships between mathematics and war, see Booss-Bavnbek and Høyrup (2003).

a *Fachkritik* in biology, physics, or in mathematics. The notion became broadly applied within the students' movement.

Fachkritik developed from a broad critique of positivism as a philosophy of science. Inspired by the work of the Vienna Circle, logical positivism claimed that science should ensure objectivity and neutrality, and that scientific investigations should be kept separated from political issues.[3] Logical positivism underwent a transformation from its programmatic formulations during the 1930s to, from the beginning of the 1950s, becoming a broadly assumed working philosophy of science. As part of this working philosophy, the normative claim about what science should do turned into the descriptive claim that science, as it is actually acted out at universities and research institutions, observes objectivity and neutrality. The *Fachkritik* reacted strongly to such an interpretation of science, claiming that positivism represents a dubious ideology rather than a proper philosophy of science.

One source of inspiration for a critique of positivism came from critical theory. In 1968, Jürgen Habermas published the first German version of *Knowledge and Human Interests*, where he presented the idea of knowledge-constituting interests. In other words, he argued that no knowledge exists that is entirely neutral. A technical interest guides natural science and technical disciplines; an interest in understanding guides the humanities; and an emancipatory interest guides the social sciences. This stipulation invited a strong critique of the social sciences to the extent they were organised in accordance with positivist principles. Assuming Habermas' (1971) terminology, positivism advocated a technical interest in general, also with respect to the social sciences. When guided by such an interest, the social sciences came to ally with oppressive forces. This observation provided a powerful departure for *Fachkritik* with respect to the social sciences.

Habermas associated a technical interest with the natural sciences, which did not motivate any *Fachkritik* related to these disciplines. Nevertheless, such a *Fachkritik* was developed profoundly, with other sources of inspiration. This observation applies also to the development of a *Fachkritik* of mathematics. An important inspiration came from

3 See Chapter 4 in this volume.

many critical investigations of mathematical modelling, on which we will concentrate in the following section.

During the same period, the notion of *kritischer Mathematikunterricht* started circulating; this is the German phrase for critical mathematics education. Initial ideas were presented by Peter Damerow, Ulla Elwitz, Christine Keitel, and Jürgen Zimmer (1974), and by Dieter Volk (1975). Soon after, Volk edited *Kritische Stichwörter zum Mathematikunterricht* (Volk, 1979), which provides a wide range of references and ideas, not only for a critical mathematics education, but for a critical conception of mathematics as well.[4] The development of critical conceptions of mathematics and of mathematics education includes much overlapping, but here I concentrate on the routes leading to a critical conception of mathematics.

Critique of mathematical modelling

Although the further formulation of a critical conception of mathematics took place in different contexts, I concentrate on what took place in Denmark. In 1972, Roskilde University Centre opened, organised according to priorities of the students' movement. In 1974, Aalborg University Centre opened with a similar profile; later it was renamed Aalborg University.

At these two universities, problem-orientation and project-organisation were implemented in all study programmes: sociology, biology, history, physics, mathematics, etc. Through problem-orientation the studies gained an interdisciplinary format. It was a period in which much educational innovation and experimentation took place, accompanied by the enthusiasm of both students and teachers.[5]

In mathematics many different problems became addressed through the students' project work. An approach often applied was to investigate real cases of mathematical modelling. Project groups investigated

4 During the same period, I started pondering what a critical mathematics education could mean in a Danish context, and in 1977 I had a PhD study approved with this ambition in mind.

5 For a presentation of how the mathematical study programmes at Aalborg University became problem-oriented and project-organised, see Vithal, Christiansen, and Skovsmose (1995). See also Jensen, Stentoft, and Ravn (2019), and Kolmos, Fink, and Krogh (2004).

such macro-economic models as the Annual Danish Aggregate Model (ADAM) applied by the Danish Ministry of Finance. Other projects addressed more theoretical economic models like the Goodwin Model that provides a possible interpretation of cyclic movements within a capitalist economy.[6] Project groups explored models applied by airlines for seat reservations, revealing that overbooking is not due to any systemic mistakes but to a carefully elaborated strategy for maximising profit.[7] The North Sea Model constructed by the Danish Institute for Fishery and Marine Research in order to maximise the fishing yield in the North Sea was examined in order to show how particular industrial interests became engraved in the mathematical structure of the model. One finds students' project reports addressing the Rasmussen Report, which is based on models that estimate an extremely low probability of a serious accident occuring at a nuclear power plant.[8]

All such project works contributed to the further development of a *Fachkritik* of mathematics. A recurring preoccupation was to identify possible political and economic interests embedded in mathematical models, thereby revealing that the postulate of mathematics-based objectivity and neutrality in such models is an illusion.

As an illustration of what such a critique might include, let us observe the book *Beskæftigelsesmodellen i SMEC III* [The Model of Employment in SMEC III], written by Mogens Niss and Kirsten Hermann (1982). Niss has been working at Roskilde University Centre from its opening and has been deeply engaged in its whole development and in supervising project works in mathematics. Hermann has worked as a secondary school mathematics teacher also with a deep concern about mathematical modelling.

1. The Simulation Model of the Economic Council (SMEC) was developed by Danish economists with the aim of advising the government and politicians about economic policy and its possible consequences. The SMEC exists in different versions,

6 For further comments on the Goodwin Model, see, for instance, Chapter 18 in Skovsmose (2014, pp. 263–280).

7 For a further discussion of such a model, see, for instance, Skovsmose (2005, pp. 79–82).

8 I supervised the group of students investigating both the North Sea Model and the Rasmussen Report. See Skovsmose (2023) for a more detailed discussion of the North Sea Model.

but Niss and Hermann concentrate on the third version and on the parts of the model dealing with employment. The *Beskæftigelsesmodel* in the SMEC III relates the level of unemployed to different economic factors, some of which can be influenced by political decisions. It is important to be able to specify as accurately as possible what could be the implications of such decisions before they are carried out. A function of the *Beskæftigelsesmodel* is to provide hypothetical reasoning to identify potential consequences of not-yet completed political actions.

2. As pointed out by Niss and Hermann, in order to develop a critical attitude towards the *Beskæftigelsesmodel*, one has not only to understand its mathematical components, but also to identify its assumptions. To construct a model, choices must be made, for instance which parameters to consider and how to integrate them into equations. There does not exist an economic reality as such waiting 'out there' to be described by a model. A macro-economic model is not just any representation of 'reality', rather it is an expression of an interpretation of economic activities and relationships. This interpretation can be guided by theoretical insights, economic priorities, political assumptions, and a range of particular interests.

3. For bringing together the whole structure of the *Beskæftigelsesmodel*, the Cobb-Douglas function of production plays a crucial role. According to this, we have $f = f(L, I)$ stating that the national product f is a function of two variables, namely the labour force L and capital investment I.[9] This function becomes used in the model-building process for identifying relevant parameters and for integrating them into equations. Simultaneously the Cobb-Douglas function creates

9 A first step in specifying this function is to stipulate that , where is an arbitrary constant. This stipulation is based on the idea that in order to increase the production by a factor , one needs to increase both the number of workers and the investment by the same factor . The function can be given particular algebraic expressions. One is , where is an arbitrary constant and is a constant between and . See also Chapter 13 in Skovsmose (2014, pp. 181–196) for comments about the SMEC III model.

a particular economic outlook by being a principal component of a classic liberal approach to economics. Through this function, an overall liberal outlook becomes installed in the way economic expertise advises the Danish government and politicians.

4. I see the critique of the *Beskæftigelsesmodel* as provided by Niss and Hermann as being exemplary in addressing not only the mathematical content of the model, but also assumptions incorporated in its mathematical structure. Such assumptions shape the space of possible recommendations that can be derived from the model. More generally, a mathematical model provides a particular description of a phenomenon, which reflects features of the model and of how it was constructed, rather than just features of what it was supposed to describe.

Critique of mathematics in action

A picture theory of language was presented by Ludwig Wittgenstein (2002) in the *Tractatus Logico-Philosophicus*, first published in a German-English edition in 1922. According to this theory, the principal function of language is to provide descriptions or pictures of reality. Furthermore, Wittgenstein claimed, at that time, that it is a formal language such as mathematics that can ensure such a picturing. An important development away from the picture theory of language became established through speech act theory. This theory highlights that language is not primarily descriptive, but performative. One does something through language. Speech act theory was anticipated by Wittgenstein (1997) in *Philosophical Investigations*, first published in 1953, two years after his death, and further developed by John Austin (1962) and John Searle (1969). The concept of speech acts was further developed in discourse theory, where the whole performative feature of language became addressed also in terms of its political ramifications.[10]

Inspired by speech act theory and discourse theory, one comes to recognise the performative aspect of mathematics. One can do things, not only with words, but also with mathematics. As one can do

10 See, for instance, Torfing (1999).

speech acting, one can do mathematics acting. By acknowledging the performative interpretation of mathematics, the scope of a critique of mathematics and of mathematical modelling gets a new profoundness. It is one thing to criticise mathematical picturing as being more or less reliable; it is quite a different thing to criticise mathematics-based actions. I aim to establish critique not only as a critique of mathematical representations but also as a critique of the actions that rely on them.[11] Here I will indicate what this could mean by referring to some examples.

Mathematics can form part of a fabrication of risks and crises.[12] Many processes get automatised by means of mathematical algorithms. This phenomenon can be observed in all kinds of economic transactions: when paying with a credit card at the supermarket, when selling and buying on the stock market. In order to operate on the stock market, one needs to make decisions, and do it fast: decisions about selling or buying, about when to do it, and about how much to trade. Any such decisions can be systematised and condensed into algorithms. This means that stock market decisions can become executed automatically. In 2006, a third of all stock market transactions in the European Union and the United States had the form of algorithmic trading.[13] That makes it possible for economic transactions to become accelerated. It can accelerate the whole situation out of balance, even with an economic crisis as a consequence. The economic crisis that took place in 2008 can be related to the mathematics that was brought into operation in the stock market.[14]

Automatic trading is just one example of mathematics brought into action in order to bring efficiency to a process. One finds automatisation in all kinds of processes, steering an aeroplane being one example. The automatic pilot can take over completely, but even when the real pilot is in control many manoeuvres are made automatically. The degree of automation gets more and more profound, and any such automatisation is made possible by a configuration of mathematical algorithms. Like any such configuration, unexpected implications can occur. As with

11 See, for instance, Skovsmose (2004, 2012, 2015).

12 For a discussion of mathematics and crises, see Skovsmose (2021). A reworked version of this text appears as Chapter 9 in Skovsmose (2023).

13 For a detailed presentation of algorithmic trading, see Johnson (2010).

14 For discussion of the relationship between mathematics and the economic crisis in 2008, see O’Neil (2016).

financial crashes, so also airplane crashes might have their explanation in some mathematical automatics going astray.[15]

Mathematics can form part of the fabrication of decisions. This encompasses all kinds of decisions, for instance decisions regarding medical issues and health care programmes. As an illustration of what this could imply, I refer to the efforts in calculating the value of a human life. Kathrin Hood (2017) points out that experts have spent 'over a century trying to develop a scientifically sound way to measure the economic value of human life' (p. 442). Whatever method the experts have arrived at, an extensive mathematical calculation is put into operation. We are still dealing with an example of mathematics brought into action.

Hood (2017) presents the following example: 'Every day, government analysts make calculations about how much human lives are worth compared to the cost of saving or prolonging them' (p. 442). One could think of the health system as forming an integral part of a humanitarian effort to save or prolong lives. However, it is also possible to look at the health care system from an economic perspective. One can think of it as making part of the government's investments. This leads one to consider to what extent the government is dealing with a good business. In order to clarify this, one needs to compare the amount of money spent on the system compared with the amount of money gained. This consideration makes it necessary to calculate the value of a saved or prolonged life. To complete such calculations, mathematics makes available a range of approaches. To me, they illustrate that mathematics fabricates decisions as well as overall perspectives that may guide decisions.[16]

Mathematics can also form part of the fabrication of possibilities. An important feature of technological development is technological imagination. Like sociological imagination, technological imagination

15 See Hawkins (2019), who raises the question whether the Boeing plane crashes can be related to automatisation.

16 The approach to identifying the value of a human life has to consider what the person would have been able to produce during the rest of his or her lifetime. A more recent approach is to consider life as a commodity, meaning that the value of a life should be identified with the price one is ready to pay for it. A recent development is found in paying attention to both the marginal costs and the marginal gains of saving a life. For a short discussion of such approaches, see Skovsmose (2021); a revised version of this text appears as Chapter 8 in Skovsmose (2023).

refers to the conception of things that do not, as yet, exist. For formulating a technological imagination, natural language might be relevant. However, natural languages have some limitations in articulating technical possibilities. Mathematics provides a different kind of language for this purpose. Whatever modern construction we might think of has been specified through a mathematical blueprint before it was constructed. This statement applies to drones, cell phones, tablets, and so on.

A mathematical conception makes it possible to identify radical new technological possibilities. As an example, one can think of the conception of the digital computer and of possibilities of digitalisation. An important step was taken by Alan Turing (1937), when he presented the abstract calculating device that later became referred to as the Turing machine. This is a mathematical conception of a computer. For doing any computing, one needs to represent an object. Within analogue computing this representation is somehow similar to the original object. In digital computing the representation is quite different: objects become represented by numbers. The whole idea of digitalisation cannot be formulated through natural language; only through mathematics can one recognise the possibility and power of digitalisation. In terms of industrial revolutions, the third one has been characterised as the digital revolution.[17] This revolution would not have been possible without mathematics.

I have restricted myself to three examples of mathematics-based fabrications; many more could be mentioned. Mathematics makes an integral part of automatisations at workplaces, and in the constructions of robots. One finds mathematics-based patterns of surveilling and controlling in all possible domains, face recognition in public places being just one example of such mathematics-based Big Brothering. Medical technologies become mathematics-based, and so do modern war machineries. Modern communication technologies are mathematics-based, and so are all security measures on the Internet. Mathematics-based fabrications can be found in all spheres of life, and with all kinds of social impact.

17 The First Industrial Revolution was characterised through the innovations of a range of new technologies. The Second Industrial Revolution refers to a phase of standardisation and automatisations of production processes.

Summary

A mathematical *Fachkritik* inspired a critical conception of mathematics. Initially such a critique concentrated on revealing that application of mathematics is not a neutral activity.

The critical conception of mathematics gained a new profoundness through a performative interpretation of mathematics. A critique of mathematics turned into a critique of mathematics-based actions, which I also refer to as mathematics-based fabrications. Such fabrications can concern any aspect of our life-worlds. There are no particular qualities associated to such fabrications due to the fact that they are mathematics-based. They can be interesting, reliable, questionable, cynical, risky, inefficient, misleading, accurate, disastrous, destructive, expensive, etc. Accordingly, a critique of mathematical performatives comes to address a range of socio-political and ethical issues.

Mathematics as a plurality of constructions

One can talk about engineering mathematics, street mathematics, applied mathematics, everyday mathematics, school mathematics, pure mathematics, any kind of Ethnomathematics. One may posit that behind this diversity there exists some kind of definitive mathematics. In fact, it is common to assume that so-called 'pure' mathematics represents *the* real mathematics.

An important idea developed along the second route towards a critical conception of mathematics is that a 'real mathematics' does not exist. Mathematics as manifest in its many varieties is a social construction, constructed in a diversity of ways. This means that 'mathematics' is an open and dynamic concept with a range of different interpretations, and new interpretations will continue to occur. There is no essence to be located within the notion of mathematics. The glorification of mathematics was based on the idea that mathematics is something unique and sublime, elevated above historical and social processes. That mathematics is a social construction removes a singularity from mathematics and its flavour of being divine.

The recognition of mathematics as a diversity of social constructions has different sources, and I will refer to three of them. First, to some philosophical observations concerning mathematics and grammar. Second, to historical observations showing that mathematics did not develop along any one-way route. Third, to ethnomathematical studies documenting the different cultural manifestations of mathematics.

A plurality of linguistic constructions

In 1939, Wittgenstein gave a series of lectures in Cambridge, where he elaborated on conceptions of mathematics.[18] As was his custom, it was for a rather closed group of people. Wittgenstein challenged such well-established conceptions of mathematics as Platonism and formalism. Turing – already well known for his presentation of the Turing machine (Turing, 1937) – joined the lectures, and he argued consistently for a formalist outlook. His presence was crucial to Wittgenstein, and once when Turing was not able to assist the lecture, Wittgenstein cancelled it.[19]

During the lectures, Wittgenstein argued against any uniform conception of mathematics, and through invented examples and thought experiments, he presented the prospect of seeing mathematics as a social construction. Furthermore, he made it possible to recognise the existence of a diversity of different forms of mathematics. In his lectures, Wittgenstein distanced himself from the conception of mathematics that he had presented in the *Tractatus*.

Wittgenstein was radical in his anti-Platonism. Rules play a crucial role in mathematics, but according to Wittgenstein mathematical rules are not based on any discoveries. There is no reality behind such rules. Mathematical rules are constructed; they have the nature of being conventions. Wittgenstein also confronted any formalism. According to formalism, a mathematical theory needs to be consistent, otherwise it has no place in mathematics. Turing insisted on this, but according to Wittgenstein consistency is not an essential requirement for a mathematical theory. What is considered consistent, and what not, depends on the rules that are put into operation – and rules could always be different. There do not exist any *a priori* rules for judging mathematical theories. We have to make do with social constructions.

18 The lectures have been collected edited by Cora Diamond and published as Wittgenstein (1989). See also Wittgenstein (1978).

19 For a mention of this episode, see Monk (1990).

One can compare mathematical rules with grammatical rules.[20] Grammatical rules become formulated and developed during history. There is nothing 'eternal' about grammatical rules. They are social constructions. Grammatical rules have some degree of permanency, and one can make grammatical mistakes by not observing the rules that are considered valid at the present period of time. Still, there is nothing ahistorical or eternal in grammatical rules.

In a similar way, mathematical rules can be interpreted as formed through historical processes, specifying ways of counting and using notions like number, point, line, and plane. There is nothing Platonic that brings validity to certain ways of using such notions. The conventions that guide their use are social constructions. Mathematical rules bring about mathematical truths, and such truths become social constructions as well. As with language, mathematics makes part of an ongoing social development. There is one more important observation to be made out of this metaphorical comparison between language and mathematics. There exist many different languages, guided by many different sets of grammatical rules. In the same way one can conceptualise the possibility of a diversity of mathematics, guided by different sets of rules.

Wittgenstein fiercely attacked dominant conceptions of mathematics. However, his philosophical critique can be taken further: it reveals that one can see mathematics as a plurality of social constructions with political coloration.[21] This observation is crucial for articulating a critical conception of mathematics.

A plurality of historical constructions

The Eurocentric presentation of the history of mathematics constitutes an integral part of the celebration of mathematics. Mathematics becomes articulated as a Western phenomenon, and simultaneously as a unique form of human knowledge. This Eurocentrism has been repeated again

20 Wittgenstein (1989) presents this idea in the following way: 'I have no right to want you to say that mathematical propositions are rules of grammar. I only have the right to say to you, "Investigate whether mathematical propositions are not rules of expression [...]"' (p. 55).

21 That mathematics is a social construction has also been highlighted by Ernest (1998) and Restivo (1992). See also Restivo, Bendegem, and Fisher (1993).

and again, for instance in many textbooks that include summaries of the history of mathematics.

A strong effort to show that the Eurocentric presentation is biased if not simply wrong, has been presented by George Gheverghese Joseph (2000), who exposes in detail the multi-cultural roots of mathematics. As a start, Joseph presents what he refers to as the 'classic' Eurocentric trajectory of mathematics. According to this trajectory, nothing of significance took place before the Greeks formulated mathematics in an axiomatic way and by doing so eliminated the empirical features in mathematical thinking. Then follows a 'Dark Age', where nothing happened in mathematics, except that Greek mathematics was carefully reproduced in the Arabic world. The 'Dark Age' became interrupted by the rediscovery of Greek philosophy, mathematics, and culture in general; and through the Renaissance and onwards mathematics developed in Europe and in 'her cultural dependencies'. Joseph describes a modified Eurocentric trajectory, according to which Mesopotamia and Egypt are acknowledged as important resources for Greek mathematics.

Joseph reveals any such Eurocentric trajectories, modified or not, as gross simplifications by outlining the very many cultural centres where mathematics developed during the so-called 'Dark Ages'. He provides a radically different picture of the historical development of mathematics than that outlined by any version of Eurocentrism. By also paying attention to what took place in India, China, Japan, Africa, and South America, Joseph documents the multiplicity of historical constructions of mathematics.[22]

Through his work, Joseph helps us to recognise that mathematics is not any unique and uniform phenomenon. It develops through complex historical processes, including many forms of interaction and communication combined with local discoveries and achievements. Mathematics appears in many different versions in different historical settings. We come to recognise that mathematics represents pluralities. The plurality of mathematics as pointed out by Wittgenstein gets an additional historical interpretation.

22 Raju (2007) argues that there are two streams of mathematics: a mathematics 'that was spiritual, anti-empirical, proof-oriented, and explicitly religious', emerging from Greece and Egypt; and a mathematics emerging from India via Arabs 'that was pro-empirical, and calculation-oriented, with practical objectives' (p. 413).

A specific issue addressed by Joseph, namely the development of calculus in India prior to its development in Europe and the transmission of that knowledge to Europe, has been analysed in depth by Chandra Kant Raju (2007).[23] Raju presents a careful analysis of the nature of knowledge transmission and the historical methods for evaluating evidence of such transmission. In addition to historical analysis, Raju argues for what he terms 'epistemological continuity' (p. 274), namely a conceptual interpretation of the available mathematical texts to seek signs of transmission of forms of thinking. Raju refers to the phenomenon of 'Hellenisation' which he describes as 'a simple trick by which a pure Greek origin was attributed to any incoming knowledge regarded as useful to Europeans' (p. 268).[24]

How could it be, then, that the Eurocentric version of the history of mathematics took such a dominant position? Important explanations are presented by Marin Bernal (1987) in *Black Athena: Afroasiatic Roots of Classical Civilization, Volume 1: The Fabrication of Ancient Greece, 1785–1985*. Let us take an extra look at the title and subtitle of the book. Bernal talks about the fabrication of ancient Greece, and he provides an account of that fabrication during the two-hundred-year period, 1785–1985. In fact, he starts his account much earlier than 1785. He covers the whole period when colonisations took place around the world, and where racism formed part of the Western outlook. By talking about 'fabrications' of Ancient Greece and not, say, about 'discoveries' of Ancient Greece, Bernal highlights that profound interpretations and re-interpretations of Antiquity took place during the colonial times. Such interpretations were guided by deep and extensive sets of preconceptions.

During the historical period explored by Bernal, it was difficult in the West to accept that Greek culture could have been influenced by the East. The whole interaction with the East as well as with Africa had to be played down in order to establish Greek culture as an integral part of Western culture. It was simply unthinkable that the White supreme European culture could have any roots in Africa or in the Orient.[25] Bernal shows

23 For a discussion of Kerala mathematics and its possible transmission to Europe, see Almeida and Joseph (2009) and Joseph (2011).

24 A particularly important example is the capture of Toledo and its library in 1085 and the subsequent translation of its books into Latin.

25 A particular version of racism took the form of 'orientalism', as coined by Said (1979).

how linguistic studies during the 1800s, not least at German universities, tried to explain away that Greek language demonstrates much influence from Eastern languages. Simultaneously, efforts were made to establish connections between Greek and German; such connections were in turn explained by a migration taking place thousands of years ago of people from the Caucasian region, with some groups moving into Greece while other groups continued into Germany. As Bernal ironically remarks, this migration seems to be the only one in history that did not leave behind a trace of broken pottery or other residues. Every kind of effort was made in order to fabricate the celebrated Greek culture as being a genuine Western culture.

The Westernisation of Greek culture also applied to the history of mathematics. When Greek culture was safely reinterpreted as being Western, Greek mathematics turned Western as well, and could be conceived of as a Western achievement. This process of 'Hellenisation', as referred to by Raju, highlights how the political power of the Christian church played a central role in this transformation.[26] Everything got together: a celebration of mathematics, a celebration of Greek culture, and a celebration of the West.

This whole worldview has to be revealed as being false, and Joseph, Raju, and others have made a huge effort to do so. The outlook is based on a wrong conception of the history of mathematics combined with layers of racism. To turn this explicit is a crucial component of establishing a critical conception of mathematics. The view of mathematics as a sublime discipline has developed together with processes of colonisation, the development of racism, and the formation of ideologies about the supremacy of European culture.

A brief discussion of the use of the expression 'Western mathematics' is warranted here: this phrase is often used by those aiming to critique precisely what they refer to in this way. However, the term troubles me for the following reason: if we look around the world today, there is nothing inherently Western about mathematics. Mathematics research occurs globally, mathematics is applied everywhere, and taught in schools as a discipline worldwide. If we go back in history, as Joseph and Raju have shown, mathematics was not exclusively 'Western' either.

26 See also Raju (2012).

So, when was mathematics fabricated as being Western? This occurred during the historical period addressed by Bernal, when colonialism and racism dominated the Western outlook, leading to a revision of the history of mathematics to fit this narrative.

A plurality of cultural constructions

More than any other research programme, Ethnomathematics has addressed the cultural plurality of mathematics.[27] In 1984, the ethnomathematical outlook was presented by Ubiratan D'Ambrosio in a plenary lecture at the International Congress on Mathematics Education (ICME-5) in Adelaide in Australia. Ethnomathematics refers to the mathematics of any cultural group. It could be Indigenous people, shoemakers, bank assistants, engineers, pure mathematicians. D'Ambrosio (2006) highlighted the importance of establishing this broader interpretation of Ethnomathematics.[28]

While Joseph provides an insight into the diversity of historical constructions of mathematics, Ethnomathematics reveals the plurality of cultural constructions of mathematics. This concretises what Wittgenstein indicated by seeing mathematics as a rule-following activity. Mathematical rules can be different like grammatical rules can be different. Mathematics can be different as languages can be different. Just as one can operate with a diversity of language games, one can operate with a diversity of mathematics.[29]

As an example of a recent contribution to Ethnomathematics, let us refer to Aldo Parra's (2018) study *Curupira's Walk: Prowling Ethnomathematics Theory through Decoloniality*. Curupira is a figure from Indigenous mythology, who walks with his feet pointing backwards, which makes it difficult to follow his route. Parra provides an extensive empirical study, and at the same time he contributes with an important conceptual development of Ethnomathematics.

27 See D'Ambrosio (1992, 2006).

28 For a recent overview of the field see Rosa, D'Ambrosio, Orey, Shirley, Alangui, Palhares, and Gavarrete (2016). Important examples of ethnomathematical studies are found in, for instance, Gerdes (2008, 2012) and Palhares (2008).

29 Explicit references to Wittgenstein's notion of language game have been made by Knijnik (2012, 2014, 2017).

Parra presents Ethnomathematics as studies of relationships. Let me illustrate what he means by that. Parra worked in a Nasa community, a group of Indigenous people living in Colombia. Among many things, he was interested in coming to understand their conception of space, and how they measured areas and distances. They had elaborated techniques for doing this, and Parra learned about these. However, he does not see research in Ethnomathematics as just identifying and describing culturally embedded notions and techniques. He sees Ethnomathematics as being concerned with relationships, which can be formed between such insights embedded in the Nasa culture and other forms of mathematics. Parra not only registered techniques applied by the Nasa community, but he also introduced alternative mathematical approaches; thus he showed how Google Maps functions.

Parra sees Ethnomathematics as 'composed of a series of contingent and purposefully constructed relations between mathematics and culture' (p. 13).[30] This could be mathematics from within a specific culture, but also mathematics that does not belong to that culture. Parra engaged himself in the educational programme developed by the Nasa community, and he also tried to contribute to this. He was not making studies *of* the Nasa people; he was doing studies together *with* them in an attempt to contribute to their environment and to meet their interests. Parra did not see himself as first of all an observer, but as a participant.

This way Parra adds a new aspect to the ethnomathematical research programme. Ethnomathematics is not first of all a study of some culturally embedded mathematical techniques and insights, it is as well a process of *involvement*. Being involved also means 'collaboration', 'participation', and 'sharing'. Such notions capture the way Parra acts out his approach to Ethnomathematics.

Milton Rosa and Daniel Clark Orey (2016) outline different dimensions of Ethnomathematics, one being the political. I find it extremely important to recognise this dimension, and Parra illustrates what this dimension might include by his involvement. Let me just indicate a different example of what involvement could mean. The Amazon Rainforest has been shrinking. It represents huge economic resources: trees can be cut and sold, and new farmland can be opened

30 In making the claim that Ethnomathematics has to do with relationships, Parra mentions that he is inspired by Alangui (2010).

up. The life conditions for the Indigenous people living there are threatened. Any ethnomathematical study engaged with Indigenous people in the Amazon needs to be involved in this drama. To try to operate as a descriptive observer means to become, not a neutral, but a cynical observer.[31]

Parra's study concerned the Nasa community. Other ethnomathematical studies address other communities. Taken together it becomes documented how different cultural contexts and practices give rise to different forms for mathematics. The ethnomathematical research programme presents mathematics as a plurality of cultural constructions. To recognise this is crucial for formulating a critical conception of mathematics.

Summary

Mathematics is formed through linguistic, historical, and cultural processes of construction, which bring about different versions of mathematics. Mathematics includes diversities. Mathematics is plural. This observation is crucial for formulating a critical conception of mathematics that does not assign any divine qualities to mathematics.

Above, I highlighted that mathematics-based actions can have any kind of qualities. This applies to any form of mathematics: engineering mathematics, street mathematics, applied mathematics, everyday mathematics, school mathematics, pure mathematics, any kind of Ethnomathematics. Whatever version of mathematics that is brought into action, also any version of Ethnomathematics, the result can be interesting, reliable, questionable, cynical, risky, inefficient, misleading, accurate, disastrous, expensive, etc. Mathematics, in whatever version we are dealing with, is in need of being critically addressed.

Mathematics as a critical resource

As already emphasised, critique not only means pointing out flaws, weaknesses, and problematic issues, but also strengths, advantages, and

31 The very idea of involvement also forms part of Knijnik's (1996) outlook. She points out resistance as being important for acting out the political dimension of Ethnomathematics.

positive qualities. In fact, a critique can point out any kind of qualities (positive or negative) of a phenomenon.

If we return to the *Fachkritik* of mathematics, as formulated during the 1970s and 1980s, one not only finds extensive questionings of mathematical modelling, but also efforts to identify models that could contribute to critical enterprises. This could be by documenting levels of economic inequalities, revealing different treatments of men and women, identifying dangers at the workplaces, showing implications of automatisation of production processes, and so on. In the initial period of the students' movements, much effort was made to ally with the general interests of workers. So, during that period, mathematics was also thought of as a possible critical resource. This possibility we will concentrate on now.

Questioning hegemonic ideologies

By publishing the article 'Critical Mathematics Education: An Application of Paulo Freire's Epistemology', Marilyn Frankenstein (1983) formulated the idea of critical mathematics education in the English-speaking context. She drew inspiration from the pedagogical ideas developed by Paulo Freire, combined with a profound critical perspective on mathematics.

Frankenstein pointed out the importance of addressing mathematics critically. This is due to the fact that forms of oppression can be masked by layers of numbers that establish an appearance of a 'necessity' of oppressive socio-economic structures. This point she formulates in the following way:

> A significant factor in the acceptance of this society's hegemonic ideologies is that people do not probe the mathematical mystifications that in advanced industrial society function as vital supports of these ideologies. (p. 327)

A pedagogy should address critically all forms of hegemonic structures, and Frankenstein highlights that mathematics also has a role to play in this critique:

> Critical mathematics education can challenge students to question these hegemonic ideologies by using statistics to reveal the contradictions (and lies) underneath the surface of these ideologies by providing

> learning experience where students and teachers are 'co-investigators' [...]. Further, critical mathematics education can link this questioning with action, both by illustrating how organized groups of peoples are using statistics in their struggles for social change and by providing information on such local groups as students may wish to join. (p. 329)

Frankenstein operates with the two main features of a critical conception of mathematics. She highlights that critique of mathematical mystifications that may function as vital support for hegemonic ideologies is essential. Simultaneously, she states that mathematics, in the form of critical mathematics education, can challenge students to question any patterns of explicit or implicit forms of oppression. Frankenstein is explicit in pointing out critical potentials of mathematics. She finds that critique is not only a reflective activity; questionings can be linked with actions. To Frankenstein critical mathematics education is not only a classroom practice; it makes part of a struggle for social justice.

In the book *Relearning Mathematics: A Different Third R – Radical Maths*, Frankenstein (1989) presents a richness of examples illustrating how a critical mathematics education can address oppressive structures. Frankenstein's work has inspired many, but I will focus here just on the work of Eric Gutstein (2006, 2016, 2018), who develops further the inspiration from Freire by showing what 'reading and writing the world' with mathematics could mean and how to combine educational activities with an activist approach.

The notions of reading and writing the world are inspired by Freire, who talked about 'reading the word' and 'reading the world'.[32] By 'reading the world', he refers to processes of interpreting social phenomena. It can be with respect to patterns of oppression that might be integrated in daily-life routines, and in this way concealed within practices that are taken for granted. By 'writing the world', Gutstein refers to processes of changing the world. This could be with respect to any kind of experienced social injustices. By seeing reading and writing the world as features of educational processes, education comes to

32 See Freire and Macedo (1987). Here Freire states: 'It is impossible to carry out my literacy work or to understand literacy [...] by divorcing the reading of the word from the reading of the world. Reading the word and learning how to write the word so one can later read it are preceded by learning how to *write the world*, that is, having the experience of changing the world and touching the world' (italics in the original).

include reflective as well as activist features. By talking about reading and writing the world with mathematics, Gutstein (2006) captures the critical potential of mathematics. It could be that mathematics forms part of hegemonic structures, as highlighted by Frankenstein, but simultaneously mathematics has a critical potential. It can be used for identifying forms of oppression and come to be a resource for political activism.

Media and racism

Racism is addressed by Frankenstein, Gutstein, and many others contributing to critical mathematics education.[33] Here I want to refer to the project *Media and Racism*, organised by Reginaldo Britto (2013, 2022) in a Brazilian context.[34]

Many statistics shows the degree of acted-out racism in Brazil. As an example, at a prominent university in São Paulo the students completing a degree in medicine are almost all White.[35] If we look at people put in jail the vast majority is Black, and the same goes for the soaring number of police killings – in 2019 in Rio de Janeiro, 1,810 people.[36] There are many such statistics that can be explored. In *Media and Racism* the focus was on making the students experience how their own observations could become expressed in numbers.[37] The project also illustrates what 'reading the world with mathematics' could mean.

The departure point for the students' investigations was the visibility of Black children in magazines circulating in Brazil. The students were divided into groups, and they were given the task of collecting photos of children shown in the magazines during a one-week period. A first observation was to see if the child was identifiable as Black or White (other racialised identities were not considered). Next, the students were asked to classify the environment in which the child was located as being either 'positive' or 'negative'. A positive environment could mean that the child was located in a wealthy and nice-looking setting, while

33 See, for instance, Davis and Jett (2019) and their chapter is this volume.
34 I have also presented Britto's projects in Chapter 2 in Skovsmose (2023).
35 See Silva and Skovsmose (2019).
36 See 'Rio violence' (2020).
37 During years, the project *Media and Racism* has been conducted by different groups of students, but in the following I concentrate on one occasion.

a negative environment could show poverty or violence. The process of classification called for a range of questions. In some cases, the classification of the child was not straightforward, and the classification of environments in being positive or negative also called for further discussions. Sometimes, the negative features were only hinted at.

Of the 41 photos that were collected by one of the groups, 36 presented White children, while 5 presented Black children. Of the White children 35 appeared in situations that were classified as positive, while only 1 appeared in a negative situation. Of the 5 photos of Black children, 3 appeared in positive situations while 2 appeared in negative. These observations became expressed through the notion of Degree of Visibility (DV).

The degree of visibility DV_i, where i refers to a particular ethnic group, is a number calculated as:

$$DV_i = \frac{\text{Number of photos with a person from the ethnic group i}}{\text{Number of photos with a person from any ethnic group}}$$

Thus, the visibilities of black children DV_b and of white children DV_w, as observed by the group in question, can be calculated as:

$$DV_b = 5/41 = 0.12$$

$$DV_w = 36/41 = 0.88$$

The numbers reflect the students' impressions: White children appear much more often in the magazines than Black children.

The DV_i provides the degree of visibility, whatever is positive or negative. However, it is possible to consider the quality of the environment, and the Degree of Negative Visibility of an ethnic group i, DNV_i, and of positive visibility, DPV_i, can be defined as:

$$DNV_i = \frac{\text{Number of appearances with negative content of the ethnic group i}}{\text{Number of appearances in total of the ethnic group i}}$$

$$DPV_i = \frac{\text{Number of appearances with positive content of the ethnic group i}}{\text{Number of appearances in total of the ethnic group i}}$$

The students could then calculate the following:

$$DNV_b = 2/5 = 0.40$$
$$DNV_w = 1/35 = 0.03$$
$$DPV_b = 3/5 = 0.60$$
$$DPV_w = 35/36 = 0.97$$

The point of making such calculations was to show that qualitative experiences can be expressed in numbers. However, such numbers are not the final words in the discussion; rather they provide a starting point for more profound explorations.

Any process of investigation calls for more investigations. We are dealing with open-ended processes. It would be natural for the students to ask, for example, if one could identify variations according to magazines. One could calculate the value of DV_i, DNV_i, and DPV_i, for Black and White children in different magazines. One could also investigate if there are changes over time. One could concentrate on photos from commercials, or on photos related to news. *Media and Racism* creates an opening for a range of further investigations. Through such activities, the potentials of operating with a notion like Degree of Visibility can become experienced by the students.

The approach used for showing the visibility of Black and White children can be used for showing the visibility of Black people and White people in any context. One can address visibility in different workplaces, different neighbourhoods, different educational settings, different political settings, different governments, etc. Furthermore, the approach can be used in relation to any classification of people in mind: women, men, immigrants, people with disabilities, etc. The very notion of Degree of Visibility and its associated calculations can be applied in many contexts. Furthermore, it is an approach whose mathematical part can be developed into a powerful tool for addressing any issue of representativity.[38]

38 For a further analysis of representativity, see Barros (2021). See also the presentation of the Bias Index in Skovsmose (2023).

Climate change

Many observations indicate that the climate of the planet is changing, and that the changes are caused by human beings. The changes are so profound that it is plausible to assume that we are entering a new historical period.[39]

Mathematics is intimately related to the discussion of climate change. A first observation is that it is impossible to talk more specifically about climate change without using mathematical modelling. In order to conceptualise climate change one needs to do forecasting. Weather forecasting is a common practice, which is now entirely based on mathematical modelling, in particular the application of dynamic system analysis. Weather forecasting has been practiced for centuries, and it existed before mathematics became applied. However, the forecasting with respect to climate changes cannot exist in any quantified form without mathematics. Mathematical climate models are tremendously complex.[40]

Through climate models, we might grasp the degrees of climate change taking place. We might also get an idea of what actions would prevent, or slow down, further changes. We are dealing with an example of *experimental forecasting*. Such forecasting is impossible without mathematics. A range of parameters forms part of a climate model, and the actual values of these parameters can be estimated through empirical observations. With changes in the values of such parameters, one might get a description of the present climate situation and how this will develop if no interventions are made. In experimental forecasting, one changes the value of some parameters in the model and observes how this will change the forecast. Through a systematic experimental forecasting one might ascertain the relevant initiative with which to respond. Experimental forecasting is crucial for formulating political recommendations for how to cope with climate changes.

Let us now return to the notions of reading and writing the world with mathematics. These expressions were used with reference to cases

39 Such a claim is made by Crutzen and Stoermer (2000), who refer to the Anthropocene as characterised by the fact that human beings are influencing the atmosphere of the earth.

40 See McKenzie (2007), Coiffier (2011), and Warner (2011).

of social injustices concerning, for instance, economic exploitations, sexism, and racism. It was also acknowledged that such readings and writings could be mis-readings and mis-writings. These possibilities are obvious when we think of reading and writing climate changes with mathematics. Any climate model might incorporate a range of assumptions, political priorities, industrial interests, also of the most dubious nature. Any mathematics-based reading and writing needs to be critically addressed.

In 'A Critical Mathematics Education for Climate Change: A Post-Normal Approach', Richard Barwell and Kjellrun Hiis Hauge (2021) discuss how climate change is addressed through mathematics.[41] Based on this discussion they provide recommendations for how to address problems concerning climate change in mathematics education. Barwell and Hauge present three groups of educational principles, which concern authenticity, participation, and reflections on mathematics.

With reference to authenticity, they recommend that we address problems concerning climate change that students find relevant in their lives; that students come to work with real data as much as possible; that the students' own ideas and values adopt a central role; and that students get the opportunity to engage in meaningful debate relating to climate change. With respect to participation, Barwell and Hauge recommend that students take part in the selection of problems, the mathematatising of problems, the selection of data, the selection of mathematical tools, and the construction of models. Furthermore, they recommend that students actively participate in their communities and get engaged in public debates. With respect to reflections on mathematics, Barwell and Hauge recommend that students are afforded opportunities to reflect on the usefulness of mathematics, but also on the limits of mathematics.

These recommendations emerged through discussion of climate change, but they are relevant whenever we try to make use of the critical potentials of mathematics. I see these recommendations as applicable to any form of reading and writing the world with mathematics. They are crucial whenever one brings mathematics into action.

41 See also Barwell (2013).

Summary

The references presented in this section illustrate that mathematics can be mobilised to work for social justice. To acknowledge this is an important feature of a critical conception of mathematics. Naturally the point is not to claim that such efforts will be successful, only that they are possible.

Formulating a critical conception of mathematics presupposes that one addresses any form of mathematics brought into action and tries to show the different qualities such applications might have. Also, any mathematics that tends to reveal and document forms of social injustices requires critique.

Critical mathematics education

A critical conception of mathematics education is formed through many contributions. The formulation of this conception cannot be related to a few people. It appears as a collective achievement.

A critical conception of mathematics challenges any understanding of mathematics as being a sublime subject representing a unique and unquestionable form of human knowledge. Instead, mathematics becomes interpreted as a powerful structure when brought into action; as a social construction incorporating linguistic, historical, and cultural pluralities; and also as a possible resource for working for social justice.

I do not try to make any distinction between critical mathematics education and mathematics education for social justice. Both approaches draw on a critical conception of mathematics. They acknowledge that mathematics can be implicated in forms of oppression and exploitation, but also that mathematics includes potentials for reading and writing the world critically. How a critical mathematics education can be put into practice is naturally an open question, but numerous attempts and reflections have been presented and explored in the literature.[42]

42 See, for instance, Alrø, Ravn, and Valero (2010), Andersson and Barwell (2021), Avcı (2019), Bartell (2018), Ernest, Sriraman, and Ernest (2015), Greer, Mukhopadhyay, Powell, and Nelson-Barber (2009), Skovsmose (2011, 2014), Skovsmose and Greer (2012), and Wager and Stinson (2012).

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