Studies on Mathematics Education and Society

BREAKING IMAGES

ICONOCLASTIC ANALYSES OF MATHEMATICS AND ITS EDUCATION

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13. How children, under instruction, develop mathematical understanding

Brian Greer

The relationship between the development and institutionalisation of mathematical understanding across millennia and its development for an individual child is the starting-point for this chapter. Greatly influenced by the writings of Hans Freudenthal, a position is taken in opposition to the theory propounded by Jean Piaget. The counterposition emphasises that a child can only be said to acquire any but the most elementary mathematics under more or less formal instruction and other forms of social and cultural interactions. The perennial debate about the relative weights that should be afforded in school mathematics to procedural competence and deep understanding is also related to the historical development of mathematics, particularly in relation to conceptual restructuring. This relationship is illustrated by the progressive enrichments of what is meant by 'number' and the basic arithmetical operations. The expansion of mathematical modelling from physical phenomena to the complexity of human interactions remains to be adequately addressed in school mathematics. And the question 'What is mathematics education for?' should be constantly revisited.

Introduction

How humanity collectively has created and systematised mathematics as a discipline is sketched in Chapter 2 of this volume. This chapter is, likewise, necessarily extremely selective. The vast literature on theory and research related to the teaching and learning of mathematics (e.g., Lerman, 2020) is minimally touched upon. The focus is restricted

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largely to the context of formal schooling (not including the tertiary level) in advanced industrial countries. The fascinating relationship between mathematics and language is barely touched upon. Many of the assertions made are offered as hopefully provocative (in the best sense of the word) speculation.

Building on Chapter 2, I attempt to elucidate the complex relationships between the development of mathematics as a project of humanity and the development of mathematics as a project for a contemporary school pupil and his/her teachers and others in social/cultural/political contexts. To provide an overview, the following key points will serve as an advance organiser:

1. *Millennia versus years.* Many have pointed to the immense challenge that is implied by expecting children to learn in a few years mathematics that took the combined intellectual efforts of humankind millennia to develop. Insofar as the individual development of mathematics happens, it happens under instruction in schools, and in other milieux, with the benefit of resources created and systematised during history, refined by evolutionary processes. Any precise correspondence between the two projects is simplistic.
2. *School mathematics should be democratic.* The number of people who become academic or professional users of formal or technical mathematics is small in relation to essentially 100% of children who attend formal schools where such exist and spend a lot of time in mathematics classes. Accordingly, it seems reasonable to recommend that mathematics education should be designed to serve the bulk of the population, while by no means neglecting cultivation of the next cadre of mathematical specialists. In pursuing this ideal, the guidance of mathematicians is obviously necessary, but far from sufficient, and sometimes obstructive.
3. *Curricular issues.* School mathematics curriculum has been showing signs of rigor mortis for decades, characterised as it is by inertia, slowness to incorporate new content and resources, dominated by the twin fetishes of algebra and calculus, permeated by premature formalisation, failure to address

the nurturing of critical attitudes towards, and agency about applying, mathematics.

4. *Intellectual rights.* Children should be accorded intellectual rights, including the right to sense-make, to receive teaching that is developmentally appropriate and conducive to understanding, and that is relevant to issues important to them, their future as adults, and their communities and cultures.
5. *The dynamic balance between homogenisation and diversity.* Within academic mathematics there is constant interplay between what Ian Hacking (2014, p. 13) calls 'unification and diversification'. In contrast to the diversity of manifestations of mathematics within cultural practices, applications and work, and everyday life, school mathematics is becoming locally and globally more homogenised, in particular due to the convergence of curricula and standardised testing.
6. *Two faces of mathematics.* Mathematics may be thought of as having two faces. On the one hand, there is the formal apparatus of pure mathematics; on the other, there is the use of mathematics in modelling aspects of reality, including physical phenomena and, increasingly for some time, phenomena involving the complexities of human life.
7. *Two places of mathematics.* Children learn mathematics beyond school, whether under some form of instruction (for example, by parents or community members) or through their own creativity when interacting with their environments, and by absorbing manifestations of mathematics within their cultures. Much more could be done to articulate the learning that occurs in the two places.

An overarching question is: 'What is mathematics education *for*?'. This cannot be separated from a consideration of the ethical responsibilities of mathematicians and mathematical educators. Mathematics education, like mathematics itself, is embedded in historical, cultural, social, and political – in short, human – contexts. And the challenge is to embrace the possibility that things can be different.

Under instruction, for understanding

> Teaching is one of the immense social influences that can affect a child, but its effects can be out of proportion to any other kind of social influence once the first beginnings of a child's life are past. In it once again knowledge builds on knowledge, but the form of experience that makes it possible is really quite unlike those forms of experience that come the individual's way when teaching is not involved. (Hamlyn, 1978, p. 144)

In my opinion, David Hamlyn's point is particularly true when it comes to mathematics teaching/learning. As argued further below, it makes no sense to posit that any child could formulate much, for example, about fractions and operations on them without instruction from others, and without a collective representational system. The *origins* of mathematical cognitive activity may be traced, as Jean Piaget has it, to reflection on actions on the physical environment, but how far can that take one? Likewise, the neuroscientists – in their study of how people develop constructs about number, in particular – seem to exhibit the same form of what might be called the foundationalist fallacy, namely that the development of any complex, multi-levelled edifice of understanding can be analysed by focusing on its beginnings.

In Chapter 2, the notion of universities and other institutions/sites as constituting constructed environments for the doing of mathematics was introduced, and the same, of course, goes for mathematics classes within schools. Following Jean Lave (1992) and many others, it will be emphasised that children learn much more than technical mathematical content in such classes. They may learn or be taught, in some general sense, to think mathematically (if they are lucky), for example to become solvers of mathematical problems in the tradition of George Pólya. They are much less likely to be taught – though it is argued below that they should be – how mathematics is embedded in human contexts; worse, they may be inculcated into harmful beliefs about the dehumanising power of numbers and equations. Unfortunately, for too many, their recollections of school mathematics are suffused with alienation and perceived irrelevance. It will also be suggested that school mathematics is instrumental in forming lasting and consequential facets of an individual's worldview, in particular relating to a simplistic view of mathematical modelling.

Researchers of mathematical teaching and learning devise their own particular constructed environments, as when, for example, an experimenter sits with a child and presents a Piagetian conservation task. Such activity represents one form of the general problem of understanding the Other. Assessment may be viewed through a similar lens, as an activity involving communication. As emphasised below, the term 'assessment' needs to be differentiated in relation to very different activity systems, from its use by the state as an instrument of control to its embeddedness as an integral part of teaching and learning. And, in general, a major issue with testing arises when the test item refers, at least on the surface, to 'real-life' scenarios, since the reactions of students, and indeed the evaluations of their responses, are affected by the degree to which the reality sketched in the item lies within the life experience of the person being tested, evaluating the test, or using the evaluations to inform their teaching.

The other emphasis implied by the title for this section reflects an aspiration that mathematics education produce 'understanding' as contrasted with superficial competence in 'pawing at symbols' (by analogy with Paulo Freire's canine metaphor of 'barking at text', which he contrasted to reading in what he considered the full sense). 'Understanding' is not so easy to define, but it is not difficult to exemplify, particularly in its absence (examples are given below).

Beginnings and continuations

> Children acquire number in the stream of their physical and mental activities, which makes it difficult for researchers to find out how this happens in detail. (Freudenthal, 1991, p. 6)

Schools have existed for a long time, but not always, and there are still societies in which 'our' form of schooling is not practiced. However, discussion here is limited to the familiar forms of schooling, to the interplay between learning in and out of school, and, in particular, to 'the poor permeability of the membrane separating classroom and school experience from life experience' (Freudenthal, 1991, p. 5).

Children develop and are taught by others before they go to school and once they are at school they continue to learn in out-of-school contexts. Consider a subset of what a five-year-old child starting school might know about *uses* of the number 5 (beyond being able to count to 5): her age is 5, perhaps represented by 5 candles on a birthday cake; everyone (essentially) has 5 fingers on each hand and 5 toes on each foot ('digits'); he may be familiar with a single coin representing the same value as 5 coins each representing 1 with the same unit; 5 spots in a pattern on a die or playing card; she may live in a house numbered 5 (between numbers 3 and 7), or travel in a bus with that number; and know that there are 5 days in the school week, 5/5 represents the 5th of May, 5.05 is five minutes past five o'clock (with the minute hand pointing to 1, representing 5), and on and on and on...

Many mathematicians (e.g., Schoenfeld, Chapter 14, this volume) report childhood insights; I can do likewise. While playing with some cardboard boxes (age five?) I found I had two boxes of different sizes, neither of which would fit inside the other. That struck me as odd, until a simple thought experiment involving a roughly cubical box and a long thin one elucidated it for me. Or take my various encounters with probability. As a child growing up some seventy years ago in a small seaside town, I had plenty of opportunities for gambling and so developed some intuitive understanding of probabilistic events (and an innoculation against gambling). Years later I was introduced to probability theory at school; later at college it was characterised as a branch of measure theory; more recently, I have written about it in relation to socio-cultural issues. So, it is possible to see what Piaget is getting at when he talks about mathematics originating in reflections on our actions. But there is a vast chasm between that and the mathematical content that even a ten-year-old is expected to engage with in school.

After a relatively short time in school, the contextual and phenomenological richness and spontaneous thinking of the child are liable to be inhibited. Further, as the child progresses through school, the disconnect referred to by Hans Freudenthal (cited above) may be strengthened through the norms of the mathematics classroom. Consider the following observation:

> It was a lesson under the heading of 'ratio and proportion' and the teacher told me that she wanted to approach the mathematical concepts in a practical way. So she offered [...] [a scenario involving mixing paints to reproduce a particular colour]. The problem seemed quite clear and pupils started to calculate using proportional relationships. But there

> was one boy who said: 'My father is a painter and so I know that, if we just do it by calculating, the colour of the room will not look like the sample. We cannot calculate as we did, it is a wrong method!'. In my imagination I foresaw a fascinating discussion starting about the use of simplified mathematical models in social practice and their limited value in more complex problems [...] but the teacher answered: 'Sorry, my dear, we are doing ratio and proportion'. (Keitel, 1989, p. 7)

Constructed environments of school mathematics

In Chapter 2, I introduced the idea of 'constructed environment' in relation to the doing of mathematics, and the same applies to the learning/teaching of school mathematics. As Lave (1992, p. 81) put it, schooling 'is a site of specialized everyday activity – not a privileged site where universal knowledge is transmitted'.

While being mindful of the distorting lenses of contemporary framings, more or less similarly organised schools have been around in many cultures for a very long time. If you stop and think about it, there is something very artificial about 'children spending large amounts of time in formal schools where their activity is separated from the daily life of the rest of the community and mediated by technologies of literacy and numeracy as well as specialized uses of language' (Cole, 2005, p. 195).

Many of the issues are exemplified very clearly in the ways in which 'word problems' (or 'story problems') are presented in school mathematics (Lave, 1992; Verschaffel, Greer, & De Corte, 2000). Children learn that there is a 'Word problem game' (Verschaffel et al., Chapter 5) whose rules include ignoring what the child knows of reality. A striking example is the following statement by a teacher in the course of a discussion with a student's mother:

> Of course, we all know that nowadays a loaf of bread costs considerably more than 21.5 francs. But after all, that's not what students have to worry about when doing algebra problems. It's the construction and execution of the mathematical expression that counts, *all the rest is décor*. (Van der Spiegel, personal communication (1997), cited in Verschaffel, Greer, & De Corte, p. 57, emphasis added)

More generally, the notion of the didactical contract between teacher and students (Brousseau, 1997) is a useful construct for describing the mutual norms that are progressively created, often implicitly, governing interactions in mathematics classrooms. By contrast with the typically implicit nature of the didactical contract, Paul Cobb (e.g, Yackel & Cobb, 1996) advocated for explicit promotion of what he termed 'sociomathematical norms'.

During their very considerable amount of time in mathematics classes, children form images about the nature of mathematics. Too often they infer from what they are exposed to that mathematics historically was the intellectual achievement of predominantly White males. They come to believe that low marks on mathematics tests are an indication of stupidity and a deserved lack of access to educational and economic opportunities. They are much less likely to form a critical disposition or sense of agency in relation to uses of mathematics. Likewise, they form images of the nature and purposes of mathematics education. For many of them, and repeatedly, when they ask the reasonable question 'Why do we have to do this?' they get the answer 'Because it will be useful later'.

Students also learn, too often the hard way, about how society constructs success and failure (Varenne & McDermott, 2018), in particular through testing. That instrument is particularly powerful in relation to attaching numbers to mathematical performance, against the background of the unreasonable political effectiveness of 'mathematics'. In the United States, a racially coded message is sent by the pervasive use of the term 'achievement gaps' when test score gaps are being referred to. Most generally, mathematics classrooms constitute constructed environments within which children learn how to fit within state systems.

Constructed environments of research on mathematical cognition

Whenever a researcher who is not the teacher engages with a child in order to try to understand that child's mathematical learning or thinking, it constitutes another very special kind of constructed environment. Here I am restricting discussion to the scenarios in which a researcher comes from outside the school and engages, typically for a short time, with students one by one. Rather than cursorily survey this vast field, I draw attention to specific aspects.

Typically, in such research, the child is asked to address a task designed by the theorist/researcher, often involving customised equipment or representations. The vast range of experiments carried out by Piaget's team constitute a familiar example, and will serve to make important points, especially through the critique of Freudenthal (1973, Appendix I, pp. 662–677). The typical experiment constitutes a very particular kind of social interaction; by analogy with the notion of a didactical contract, the idea of an experimental contract may be invoked. It is necessary to consider how the children in this situation construe what is going on, why they are there, what is required of them. In my experience and observation, such considerations are often minimally addressed by experimenters.

Aligned with this framework, Freudenthal (1973) very strongly criticised Piaget's insufficient attention to the language used and whether or how the child understands it; a parallel may be drawn with the role of communication in assessment (see below). Experiments on conservation, for example, are particularly open to this kind of scrutiny and a range of experiments has shown that altering the experimental contract or the nature of the communication in apparently minimal ways can have a marked effect on the responses (see, e.g., Donaldson, 1978).

Most seriously, we may ask the general question: How does the experimenter/theorist know that the design and presentation of the task, and the children's responses, constitute an appropriate test of the constructs embedded in the theory? Is it possible that the experimental tasks and communications, consciously or unconsciously, are designed to support rather than test the theory? That is an extremely serious charge that deserves to be taken seriously. For example:

> By a suggestive design of the experiments it is achieved that the subjects reconstruct a landscape according to the Piaget theory of multiplication of relations, that is by means of a Cartesian coordinate system. (Freudenthal, 1973, p. 669)

To return to the point made in the opening quotation, in relation to conservation of volume, I would be much more impressed by a report of two children being poured lemonade from identical bottles into differently shaped glasses and one of them objecting that it was unfair.

'Assessment'

> If you want to sort people, make them run a race; if you want to see if kids can 'do it', then give them adequate time to 'do it'. (Stage, 2007, p. 358)

For a long time, I have found it problematic to use the same word to refer to two very different families of practices. One family, the focus of this section, involves producing measures that allow students, and groups of students, to be measured and ranked, often with high stakes attached. Another family has to do with interacting with the student in order to form conjectures about the student's understanding, cognition, beliefs, and so on; as such, it is an integral part of teaching/learning.

Following these introductory remarks, I consider just three from the vast range of relevant aspects: the analysis of assessment in terms of communication; the diversity of social realities in relation to attempts to include mathematical modelling in test items; political issues in the uses of testing for purposes of the state.

Assessment as communication

Concentrating on the communicational functions of assessment affords pointed contrasts between the two activity systems distinguished above. In general, assessment involves: communication to the student about a specific competence to be demonstrated through a particular task; action by the student in an attempt to demonstrate the required competence insofar as they understand it; some form of evaluation of that attempt; and communication of the interpretation of that evaluation to the teacher and others. In these terms, a standardised written or computer-administered test may be seen as extremely impoverished in terms of communication at every stage, particularly when there is no opportunity for clarification through subsequent iterations of communication. 'In short, the typical written assessment is closed in terms of time, in terms of information, in terms of activity, in terms of social interaction, in terms of communication' (Verschaffel, et al., 2000, p. 72).

To take a simple contrasting example, a teacher may ask a student to subtract 17 from 24 and the student might give the answer '13'. The teacher may conjecture that the student has exhibited the 'subtract the smaller from the larger within any column' misconception and ask further questions to test this conjecture. Finally, the communicative act of evaluation may consist of much more than a simple statement that the student's answer was wrong, but be combined with an explanation of why, and of how that misconception can arise.

Assessment and modelling

Test items often resemble word problems in presenting a description of a real-world situation that the assessed is expected to interpret and model mathematically. In the absence of open communication, it then often happens that the model depends on the life experience of the generalised modeller as well as that of the testee, as in the following example discussed by William Tate (1995, p. 440):

> It costs $1.50 each way to ride the bus between home and work. A weekly pass is $16.00. Which is the better deal, paying the daily fare or buying the weekly pass?

It should be obvious that assumptions made (e.g., that work occurs five days a week) will affect a person's interpretation and response and that the person's form of life will influence the assumptions made. There is no 'right answer' and if it is assumed that there is, and there is no opportunity for clarification, the item is accordingly unfit for assessment. In particular, as Tate (1995, p. 440) pointed out, 'the underpinnings of school mathematics, assessment, and pedagogy are more often closely aligned with the idealised experience of the White middle class'. More generally, testing may be seen as an instrument of cultural violence, as when 'test-score gaps' are mislabelled as 'achievement gaps' with no qualifications as to how achievement is defined.

Testing as an instrument of the state

Above all, 'assessment' is a political issue. Episodes of recent history within the United States in terms of clashes between political, corporate, and educational goals are analysed by Alan Schoenfeld (Chapter 14, this volume), a battle-scarred veteran of many campaigns. A key point that he makes is that another communicative function of testing is to convey to teachers and students what is expected, encapsulated in the acronym 'WYTIWYG' (What You Test Is What You Get). Schoenfeld illustrates from his experience how the ways teachers teach are liable to be distorted under the pressure of upcoming high-stakes tests. The politics of global testing are analysed by Paola Valero and Lisa Björklund Boistrup (Chapter 15, this volume) and Mark Wolfmeyer (Chapter 16, this volume).

The goal of understanding

> Most people have been taught mathematics as a set of rules of processing – an agreeable experience when they have learned to master them, and a disagreeable one if they have failed. (Freudenthal, 1991, p. 3)

Later, Freudenthal argues that elementary arithmetic cannot be learned *other than through insight*, but as the school student progresses to more advanced mathematics, 'the learner's insight tends to be superseded by the teacher's, the textbook writer's, and finally by that of the adult mathematician' (Freudenthal, 1991, p. 112).

The section title expresses an aspiration that mathematics education should produce 'understanding', something that is difficult to define but easy to illustrate, particularly in its absence. A simple example comes from *Productive Thinking* (Wertheimer & Wertheimer, 1982/1945). Children were asked (p. 130) to find what number the following expression is equal to:

$$\frac{274 + 274 + 274 + 274 + 274}{5}$$

A child who correctly computes a repeated addition, or multiplication, followed by a division, demonstrates computational fluency, but such a performance surely suggests a lack of understanding. The authors related his surprise that, while most of the bright students he asked 'enjoyed the joke' (p. 112), 'a number of children who were especially good at arithmetic [...] were entirely blind' (p. 113).

As a second example, I posed this question to future elementary school teachers studying slope as represented on graphs:

> A candle, initially 24 cm high, is burning *down at the rate of 3 cm per minute*. If you plot the graph of height of the candle (in cm) against time (in minutes), what will be the slope of the line?

Most of the students demonstrated competence by plotting the line and calculating the slope; hardly any showed understanding by pointing to the answer (-3) given in the italicised part of the question.

A distinction may be drawn between 'internal' understanding and 'external' understanding. The former refers to making connections, noticing and exploiting structure, within 'disembedded' (Donaldson, 1978) mathematics, as in the example from the Wertheimers' book cited above. The second example is about articulating procedures (plotting points and calculating slope) as opposed to understanding that slope of a straight line corresponds to a constant rate of change in some variable.

The relationship between procedural competence and conceptual understanding is central in discussions on mathematics and mathematics education. The problem, as suggested in the opening quotation, arises when procedural competence dominates (as it does in communicationally impoverished forms of testing).

Learning from history

> We know nearly nothing about how thinking develops in individuals, but we can learn a great deal from the development of mankind. (Freudenthal, 1991, p. 48)

> It is with children that we have the best chance of studying the development of logical knowledge, mathematical knowledge, physical knowledge, and so forth. (Piaget, 1970, pp. 13-14)

The first obvious comment is that these quotations illustrate the chasm between the positions of Piaget and Freudenthal, as reflected below. Studying the history of mathematics is extremely difficult to do for many reasons, and it has often been done very poorly, as Jens Høyrup, for one, has made clear (Greer, 2021). The central question in this section of the chapter is: 'What guidance for mathematics education can be derived through studying the history of mathematics?'. Many have pointed out that children in school are expected in a few years to come to grips with mathematics that took humankind millennia to develop:

> School is seen as a magical shortcut that allows ideas arduously developed by humanity over thousands of years to be transmitted in a few years to a random human being. (Hofstadter & Sander, 2013, p. 391)

To the extent that those ideas can be transmitted, how is it possible? The short answer is that it is achieved predominantly through instruction in a constructed environment.

The position taken here is that any kind of simplistic version of 'ontogeny recapitulates phylogeny' (Gould, 1985) is untenable. The complexity of the interactions between biological and cultural evolution must be addressed (Cole, 2005). Some general comments on Piaget's treatment of mathematics within his theory of genetic epistemology are followed by a specific critique of the supposed correspondence he claimed between the 'mother structures' of Bourbaki and constructs within his theory of cognitive development. A contrasting position is based largely on Freudenthal's (1991) conception of 'guided reinvention' and James Kaput's (1994) conception of 'applied phylogeny'.

Among the facets of the cultural environments in which children grow up is the panoply of representations, both formally introduced within mathematics classes and encountered in the environment in general. Assimilation/accommodation of existing, collectively sanctioned, representations is a very different matter from the original slow development, with evolutionary selection, of those representations. Another glaringly obvious historical observation, evident through a cursory glance through Florian Cajori's (1928) painstaking work, is that notations and representations are contingent, arbitrary, underdesigned – whether that will ever change is doubtful.

Finally, in this section, to illustrate some of the issues, I take a look at a particular content area, that of negative numbers.

Simplistic parallelism: Piaget and Bourbaki

> The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress made in the logical and rational organization of knowledge and the corresponding formative psychological processes.
> (Piaget, 1970, p. 4)

In this section, I first offer some general assertions about the significance of Piaget for the study of mathematical cognition and the practice of mathematical education, then I specifically discuss Piaget's claim that the mother structures of Bourbaki correspond to empirically supported developmental cognitive structures.

Piaget identified himself primarily as a genetic epistemologist, not a developmental psychologist. He became the latter in service of the former, arguing, as cited earlier, that since little is knowable about the origins of human thinking, the best recourse is to study children. Moreover:

> The defining feature of Piaget's approach [...] is that the stages and mechanisms that he postulates are not psychological, or historical (so he is not 'reporting' an accidental parallel between the two), but rather, epistemological – this is how knowledge is inherently constructed. (Kaput, 1994, p. 84)

Rather than attempt a systematic critique of this position – an enormous undertaking – I merely state my conviction that I find it unconvincing, unless it is reduced to the banal statement that knowledge grows through developmental processes which can be described in such general terms as to fit both domains. I would even conjecture that having framed his position on intellectual-aesthetic grounds, Piaget devoted much of his life as an experimentalist to confirming it.

Further, while in the spirit of making controversial statements, I will suggest that in his emphasis on adaptation, initially with respect to the physical environment and originating in his first experimental investigations as a biologist studying adaptation of molluscs, Piaget extended, in a kind of metaphorical way, to the other environments that I have labelled cultural and constructed (educational). In contrast, it has been argued that

> the contemporary study of the role of culture in human development is hampered by the continued failure of behavioral scientists to take seriously the co-evolution of phylogenetic and cultural-historical change in shaping processes of developmental change during ontogeny. (Cole, 2007, p. 236)

Piaget's monumental oeuvre is of great importance, especially against the backdrop of behaviourism in the earlier part of the twentieth century (space does not allow consideration of the part played by Lev Vygotsky and other Russian psychologists in theorising the social, collective complexities of education, nor the intimate involvement of Russian mathematicians in school mathematics). At a macro-level, Piaget revealed the complexity of children's thinking; however, there are several criticisms that are particularly relevant to mathematics education. Of these, perhaps the most important is that, in postulating an explanation in terms of adaptation to environment, he understated the differences between physical, social, political, and constructed environments. Having spent a reasonable amount of time studying his work, and doing related research on children's cognition, I, like others, find the claim unconvincing that people, whether historically and collectively, or contemporarily and individually, construct mathematics through reflection on their interactions with the environment:

> The epistemological approach which starts from the position of the individual alone is so wrong. The fact that such an approach fits in with the biological approach which similarly considers the individual organism in relation to its environment equally shows the inappropriateness of that as a model on which to construe the growth of knowledge and cognitive development generally. (Hamlyn, 1978, p. 59)

Piaget believed that the systematisation of (some parts of) mathematics by Bourbaki, in particular their postulation of three 'mother structures' constituted a striking confirmation of his position. It is my impression that the Bourbaki group was willing to collude in this belief as it strengthened their own case to be central to a network of structuralist ideas. The strongest critique of the supposed relationship was made by Freudenthal, who wrote:

> Poor Piaget! He did not fare much better than Kant, who had barely consecrated Euclidean space as a 'pure intuition' when non-Euclidean geometry was discovered! Piaget is not a mathematician, so he could not know how unreliable mathematical system builders are. [...] Mathematics is never finished – anyone who *worships* a certain system of mathematics should take heed of this advice. (Freudenthal, 1973, p. 46, emphasis added)

Piaget (1970) did acknowledge the emergence of category theory as a systemic reformulation, but without suggesting how that affected his correspondence hypothesis.

Another mathematician who tangled with Piaget was René Thom, a topologist known for his development of catastrophe theory. In particular, Thom argued that Piaget's position on geometry was gravely wrong, as did Freudenthal: 'it is a serious mistake if, to justify a particular kind of didactics, people tell you Piaget proved Euclidean geometry to start psychologically with Cartesian coordinates' (1973, p. 669).

To close, a particular issue that has puzzled me for nearly forty years is the ontological status of 'cognitive structures' as something that a child 'has' (Jeeves & Greer, 1983, pp. 65–69). In those pages, we used a lengthy quotation from Feldman and Toulmin (1976, p. 426), which seems to go to the heart of the issue:

> Nowhere, it seems, are the differences between the problems involved in formally representing a theory and the problems in empirically testing it so difficult to keep separate as in the area of cognition. Just because the theoretical system in question can plausibly be represented as corresponding to some mental system in the mind of an actual child, we may be led to conclude that the formalism of the theoretical system must be directly represented by an isomorphic formalism in the mind of the child... In this way, ontological reality is assigned to the hypothetical mental structures of the theory simply on the basis of the formal expressions by which they are represented in the theory.

Guided reinvention, applied phylogeny

> Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now. (Freudenthal, 1991, p. 48)

The qualification within the statement is crucial; Freudenthal's vision of reinvention was with strategic instruction, guided by what Kaput (1994, p. 83) termed 'applied phylogeny'. Kaput introduced this term with appropriate warnings about the cognitive appeal of 'ontogeny recapitulates phylogeny' including 'the differences between a collective historical enterprise and an individual's learning' and 'the irregularity of historical developments'. To use an obvious example, nobody would suggest that children should be taught the Roman way of labelling natural numbers before the decimal system.

By way of example of 'repeating the learning process of mankind', consider multiplication and division of positive real numbers. In ancient Mesopotamia (as extensively documented by Høyrup) and in Peru (Urton, 1997), for example, the operations were linked polysemously to cultural practices. For the Quechua of Peru, multiplication and procreation were closely connected (and remember that, in the Bible, people are told to 'go forth and multiply'). To put it another way, multiplication and division can be used to model many classes of situation. In particular, there is a marked contrast between 'asymmetrical' situations – in which the quantities multiplied are clearly distinguishable as multiplicand and multiplier – and 'symmetrical' situations, such as rectangular area, in which they play equivalent roles. As a consequence, in the former case, there are two distinct forms of division, but not in the latter (Greer, 1992).

Contrast the above with the formal treatment of the operations. From a Bourbakian perspective, they are applied in decontextualised computation and organised within groups and other disembedded structures. In schools, arguably a great deal too much emphasis is on computation and formal properties such as commutativity, treated abstractly and not in relation to situations modelled, within which its nature varies greatly – sometimes addition and multiplication are trivially commutative, sometimes not. Similar comments apply to the statements that addition and subtraction, multiplication and division are inverse operations.

The contrast between an abstract structure, such as a group, and diverse instances of it, such as transformation groups in geometry, was characterised by Freudenthal (1991, p. 20) in terms of 'rich' and 'poor' structures. Groups, historically, were encountered as rich structures in multiple different contexts and only axiomatised relatively recently. In Freudenthal's vision for teaching mathematics, the axiomatisation should come as the culmination of a long process – starting with the axioms or the poor/pure structure was, in his view, a gross pedagogical error, a 'didactical inversion'. Thus (p. 29) 'the didactically recommendable direction will be the same as that in which mathematics arises, that is, from rich to poor'.

In Freudenthal's vision, also, he emphasised changing the view of mathematics learning as accumulating content and 'neatly tailored abilites, the mastery of which can be tested "objectively" (as they call it) – that is, by computers' (1991, p. 49) to experiencing important mathematical activities:

> The learner should reinvent mathematising rather than mathematics; abstracting rather than abstractions; schematising rather than schemes; formalising rather than formulas; algorithmising rather than algorithms; verbalising rather than language...

Cajori (1898) wrote that: 'The history of mathematics may be instructive as well as agreeable; it may not only remind us of what we have, but also teach us how to increase our store'. Citing Augustus De Morgan, he continued: 'The early history of the mind of men with regard to mathematics leads us to point out our own errors; and in this respect it is well to pay attention to the history of mathematics'. This principle lies at the heart of Kaput's notion of applied phenology. Painstaking research in the historical record can identify cognitive obstacles and the ways in which they were, often after considerable time, resolved. That can then guide the teaching of children, in line with the quotation at the start of this section.

Material representations

> A class in arithmetic [...] will find it astonishing that it should have taken so long to invent a notation which they themselves can now learn in a month.
>
> (Cajori, 1928, p. 3)

The importance of representations in the growth of mathematics historically is discussed in Chapter 2. There are, of course, huge differences between, on the one hand, the invention by mathematicians of representations in the service of the mathematics symbiotically being created and systematised, and, on the other, the presentation to children of evolutionarily stable representations. Is it any wonder that the presentation of the products of such long-drawn-out efforts as off-the-shelf resources for children to use is rife with complications? For example, mathematicians are prone to regard the graphical representations of functions in the Cartesian plane as perspicuous, yet a mass of empirical evidence exists to show that misinterpretations are extremely common and difficult to dislodge.

In passing, a point that is obvious to anyone reading Cajori's (1928) painstaking survey is that mathematical vocabulary, notations, and representational conventions are created very arbitrarily (an example being the conventional use in algebra of a, b, c as parameters and x, y, z as variables). Why do children (in English, as in many other European languages, but not German) have to deal with 'isosceles' rather than the Anglo-Saxon 'twesided' used by Robert Recorde in the sixteenth century (Cajori, 1922)? For a discipline whose exponents pride themselves on their rationality, the representations used in mathematics are surprisingly user-unfriendly.

One way in which design *has* been prominent is in the conscious development of material teaching/learning resources termed 'manipulatives'. The earlier history of these in (some parts of) European mathematics education is well covered in De Bock and Vanpaemel (2019). Reflecting a distinction that is very clear for computer representations (Kaput, 1992), these are representational resources which children can use for recording, but also for acting upon. For teaching arithmetic, examples include Cuisenaire rods and the multibase arithmetic blocks designed by Zoltán Dienes.

The prominence of manipulatives has declined. One reason is that their pedagogical effectiveness has been called into question. Rather like the problems in trying to turn Pólya's heuristics into classroom gold the issue is that you cannot understand how to exploit a heuristic such as 'think of a related problem' unless you know a great deal already about what a related problem looks like. Likewise, those who can understand how a manipulative relates to the mathematics it is designed to illuminate have little need for the manipulative. Conversely, manipulatives may be of limited effectiveness for those who do not understand the connections (like the child who told Kath Hart that 'bricks is bricks and sums is sums'). With respect to the multibase arithmetic blocks:

> Children who already understood base and place value, even if only intuitively, could see the connections between written numerals and these blocks [...] But children who could not do these problems without the blocks didn't have a clue about how to do them with the blocks [...] They found the blocks [...] as disconnected from realty, mysterious, arbitrary, and capricious as the numbers that these blocks were supposed to bring to life. (Holt, 1982, pp. 281–219)

New representational windows

> [...] information technology will have its greatest impact in transforming the meaning of what it means to learn and use mathematics by providing access to new forms of representation as well as providing simultaneous access to multiple, linked representations. (Kaput, 1986, p. 1)

I seem to remember Benoit Mandelbrot, speaking in 1992 saying that 'the computer has put the eye back in mathematics'. In evolutionary perspective, the computer age represents a new epoch in the creation of representational resources, with consequent massive implications for cognition (Kaput & Shaffer, 2002). Examples follow.

One of the first such revolutionary visions exploiting developments in information technology (IT) was the language Logo, designed by Seymour Papert. It rivals the Turing machine in terms of the simplicity of its primitives. Inspired by the young Papert's fascination with gears, the basic mechanism (literally embodied in drawing machines called 'turtles' which drew geometrical configurations controlled by the language) was an axle with equally sized wheels on the ends, which turn at the same speed either in the same direction, thus moving the turtle forward a stipulated distance, or in opposite directions, rotating it through a stipulated angle. The second element in generativity is the programming language in which the user can define and name routines; the names are then appended to the language. Logo produced a way of conceiving planar geometry very differently from Euclidean geometry. A circle, for example, is approximated to any desired degree of precision, as a regular polygon.

Another geometrical system, more closely linked with traditional geometry is Geometric Supposer (GS) (see also Cabri, and 3-D versions). Within GS, constructions can be defined similar to those of Euclidean geometry and recorded as *procedures* rather than *drawings*. The theorem that if you construct any quadrilateral and join the midpoints of the four sides you get a parallelogram takes on a different feel when you can grab it by a vertex and make the whole construction waltz on the screen. Among many other wonderful creations may be mentioned Fathom, which affords exploration of probability and statistical modelling.

Kaput himself designed SimCalc and other software as resources for teaching calculus exploiting the kinds of features that he analysed (Kaput, 1992). And STELLA makes dynamic system modelling accessible to high school students (Fisher, 2021).

Reading such work in the 1990s, a reader might well have thought: 'Just think what they'll be able to do in schools thirty years from now'.

They would be greatly disappointed. There are many reasons for this, some of which are to do with the IT industry seeing more profit in other kinds of product than in tackling the complexity of teaching mathematics. Another major reason is the failure to grasp the need for teacher training and to provide the necessary support. For example, Papert's vision of Logo as a mathematical world in which children could learn by themselves was arguably overoptimistic, and it progressively became clear that its effectiveness could only be realised under the guidance of skilled teaching.

A historical example: Directed numbers

> 3 – 8 is an impossibility, it requires you to take from 3 more than there is in 3, which is absurd. (De Morgan, 1810/1931)

> Minus times minus makes a plus.
> The reason for this we need not discuss. (Attributed to W. H. Auden)

The case of directed numbers may be taken as paradigmatic for considering how a study of the history of mathematics might inform contemporary teaching. De Morgan was an eminent mathematician but balked at an arithmetical expression that quite young children today are expected to take in their stride. He was right if the only interpretation of $n - m$ (where n and m are whole numbers) is the removal (in some sense) of m countable entities from a set of n. (And he did acknowledge the *algebraic* interpretation of $n - m$ when $m > n$.)

It could be argued that there is a fundamental epistemic shift illustrated here, from $n - m$ as a direct representation of a situation (taking objects away from a set of objects) to $n - m$ as a mathematical expression that can be used to model many situations – such as bank balances (did De Morgan never get into debt?) or scales with a zero such as those for measuring temperature or altitude relative to sea level. (When children later are being taught multidigit subtraction, e.g. 43–18, the teacher may say something like 'you can't take 8 from 3, so you borrow 10 and take 8 from 13'). From a purely structural point of view, the expansion of the positive whole numbers to all integers means that subtraction is closed over the set of whole numbers, which form a group under addition, with identity element 0.

Ademio Damazio (2001, p. 209), on the basis of classroom observations of children being taught about directed integers, concluded that 'the students did not overcome the concept of number as an ability to count concrete objects instead of as abstract objects that can be operated independently'. Well, why would you expect them to achieve such a feat over a series of twenty class lessons, what De Morgan, after a full mathematical education and career, failed to achieve? But try telling that to curriculum developers!

On the basis of observations of a teacher, Damazio (2001, p. 208) commented that 'the teacher ceases to evidence relevant aspects for concept formation. You can do that if you are concentrating on calculative fluency alone. In particular, the notion of a relative zero (as a reference point) as opposed to that of absolute zero [...] is the foundation of the concept of relative whole numbers'.

The case of multiplying and dividing negative numbers is much more complex than addition and subtraction and took even longer to resolve to the satisfaction of rigour-demanding European mathematicians (with false starts over centuries along the way, and eventual survival of what works). Within this context, the shift to modelling is even clearer. How can the plausibility of the rule Auden was told not to discuss (see above) be communicated to a child? There are a number of general approaches:

- *Patterns*. A two-dimensional table can be constructed with 0, 1, 2, 3, ... along each axis and the products in the body of the table. Extending back along each axis to –1, –2, –3, ... and following the patterns makes the rules for multiplying directed numbers at least somewhat plausible (for an excellent discussion, see Sawyer, 1964, pp. 297–300) and a similar exercise can be carried out graphically (pp. 300–309). (Such patterns could be thought of as 'localised structures', partial reflections of the structures of Bourbaki and the like.)
- *Modelling linear change over time*. Suppose you are walking on steps at a constant rate of n steps per minute, not up $(+n)$,

but down $(-n)$. Then t minutes earlier $(-t)$ you were nt steps higher than you are now.

- *The algebraic/geometrical approach of the Babylonians*. Consider the expression $(x-a)(y-b)$. It is straightforward if $x > a$ and $y > b$ and easily verified through examples that its expansion as $xy + x(-b) + (-a)y + (-a)(-b)$ 'works' if interpreted as $xy - xb - ay + ab$. And there is a geometrical counterpart.
- In Greer (2005), I cited a hilarious formal 'proof' written by mathematicians for sixth-grade students and I cannot resist reproducing the start of it here:

> First, if a number a satisfies $b + a = 0$; then a is $-b$. That is how $(-b)$ is defined, as the additive inverse of b. Second, $N \times 0 = 0$ for any number N because the area of a rectangle with one side zero is zero [...] Third:
> $$0 =(-1) \times 0 = (-1) \times (1 + (-1) = (-1) \times (1) + (-1) \times (-1)$$
> (California Department of Education, 2000, p. 144)

(How I like mathematicians to speak of 'a rectangle with one side zero'!).

- *A student's pragmatism*. I asked students in a general college mathematics class to (a) say if they believed (–1) x (–1) = +1 and (b) explain why. The answers were, as you might expect, mainly appeals to authority of some kind. However one student wrote that he believed it because every time he had operated according to that belief in a test he had gained marks, and conversely.

I would be prepared to argue for what would generally be considered a radical solution, namely to postpone treatment of multiplication and division of directed numbers until college, at which point it could be treated with the formal and informal thoroughness it warrants with students who have more relevant experience. I can cite one prominent mathematician who wrote that 'the multiplication of negative numbers (like the addition of fractions) can and should be postponed' (Hilton, 1984, p. 8). Whenever it is introduced, it damned well ought to be discussed.

What is mathematics education for?

Introduction

> Mathematics as an aim in itself [...] is an important aspect, although of less concern to us here, since our subject of mathematics education embraces a much larger group than only future professionals of whom once again only a small minority choose mathematics in itself. (Freudenthal, 1991, p. 3)

In the above quotation, Freudenthal expresses clearly a theme that is omnipresent in the following discussion. Think of a pyramid representing the population of those who spend a lot of time in mathematics classes at school – in most of the world, essentially everyone. A very small section at the top then corresponds to those who will constitute the next cadre of mathematics researchers and tertiary level teachers of mathematics. A larger section below that represents those who use significant amounts of mathematics in their work – scientists, engineers, (some kinds of) social scientists and, generally, certain specialists within most fields (though there is considerable research showing that architects, for example, may use little of the formal mathematics of which they have been required to show mastery (Hacker, 2016)). The remaining bulk of the pyramid represents everyone else. Quite simply put, the thrust of this section of the chapter is to argue that the pyramid should be inverted, so to speak, so that school mathematics much more deliberately reflects the needs of the majority; to put it provocatively, school mathematics education is too important to leave to mathematicians who are primarily invested in perpetuation of their subspecies.

As a start, I pick up on earlier discussion of how formal mathematics influences curriculum, a particular case being the impact of Bourbaki. While the overt influence of Bourbaki has waned, its ghost still haunts mathematics classrooms (along with those of its extended family) in terms of premature formalisation. The Common Core State Standards used in the United States may be taken as a representative contemporary curricular design in terms of content that could be termed 'Bourbaki light' – a framework based on progressive mathematical structuration with premature formalisation rather than pedagogical and developmental considerations, and with scant attention to long-term pre-emptive planning or to the critical points at which conceptual change must be carefully nurtured.

Accordingly, a counter-position is presented whereby curricular design is fundamentally respectful of the child's capacity for understanding and accumulated experience at any point. The first point to be made is that given the vast amount of recorded and systematised mathematics, the selection problem (already mentioned in relation to Bourbaki) rears its head. In the face of what might be considered the (somewhat) reasonably reactionary nature of curricula, a number of radical alternative design principles are proposed, in particular aimed at making school mathematics useful to the adults that students become, rather than being a preparation for a small elite. As part of that argument, I contend below for substantial reductions in the level of formalisation of content and framing (which would also help teachers). These proposals are also linked with the proposal to move the centre of gravity, substantially, away from technical mastery and towards understanding – which, it is argued, would benefit also the elite who become advanced students of mathematics (and, again, teachers).

Given the degree to which mathematics formats our lives (to use Skovsmose's term), those who frame school mathematics now have a responsibility to include instruction about modelling, its purposes, and its limitations. These aspects arise sharply, and very early on in elementary school, in the context of 'the bizarre genre of word problems', a locus within which quite young children could be taught to distinguish between modelling that is precise, modelling that is a more or less good approximation, and modelling that is plain wrong.

All of these considerations build to the argument that the relationships between mathematics and the social sciences be re-examined (see Chapter 9, this volume). Particularly important aspects include:

- The nature and purposes of mathematical modelling.
- Talking with students about mathematics, what it is *for*, how it is taught/learnt, the cognitive obstacles, its history, and its political ramifications.
- Mathematics in relation to aspects of life important to the students, their families and communities.

Premature formalisation

The influence of Bourbaki on mathematics and mathematics education in the twentieth century is discussed in Chapter 2. While the overt influence of Bourbaki and other formally-intensive statements by mathematicians has waned, its lingering influence can be seen in the perseverance of premature formalisation.

By way of an example, the Common Core State Standards for Mathematics (CCSSM) developed in the United States may be taken as a reasonably representative curricular framework in terms of specifying mathematical content. It is critically flawed in many ways, in particular in its failure to offer pedagogical guidance. In his masterly comparative analysis of national curricula in fourteen countries, Geoffrey Howson (1994, p. 26) made the crystal-clear point that 'a curriculum cannot be considered in isolation from the teaching force which must implement it'. I argue below that the dominance of mathematicians in its framing illustrates the harmful effects that mathematicians can have on school mathematics education.

Unlike some manifestations of New Math of the 1960s (which I remember living through), set theory is not proposed as the starting point for children's learning of mathematics, thus avoiding the absurdity of confusing the foundations of mathematics education with the foundations of mathematics as traditionally presented by philosophers; nevertheless, premature formalisation is pervasive. The ghost of behaviourism lurks, in that the framework is very much presented in terms of an incremental progression on a superficial metric of complexity, logical in the sense of the adult mathematician's retrospection, not in terms of children's ability to understand.

As an example, consider the extension of multiplication and division beyond the natural numbers. This is what we read:

> **Apply and extend previous understandings of multiplication and division to divide fractions by fractions.**
>
> Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g. by using visual fraction models and equations to represent the problem. For example, create a story context for $(2/3) \div (3/4)$ and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that $(2/3) \div (3/4) = 8/9$ because $3/4$ of $8/9$ is $2/3$. (In

> general, $(a/b) \div (c/d) = ad/bc$. How much chocolate will each person get if 3 people share ½ lb. of chocolate equally? How many ¾-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length ¾ mi and area ½ square mi? (CCSSM, p. 42, for Grade 6)

If you are not laughing, you have not been paying attention. It is hard to expunge the image of someone embarrassedly saying to a guest 'I am so sorry, I can only offer you 8/9 of a serving of yoghurt'. Another compelling image is of the unfortunate person tasked with devising a believable story to go with $(2/3) \div (3/4) = 8/9$ for a sixth grader.

Indeed, the ways in which fractions are treated are indicative of the problems I am trying to elucidate. Here are some of the issues:

- Essentially nobody apart from children in school needs to compute something like $4/7 + 5/11$. People like carpenters and engineers, who make things that work, use decimals or binary fractions. What would be lost by following their example, restricting instruction to the few fractions and uses of fractions that people generally find useful (as approximations, for example)? The loss for formalists would be the lack of the formal closure of the positive rationals under the four arithmetical operations.
- Not unrelated is the common observation that fractions often constitute the first wall of incomprehension in mathematics class. A *Peanuts* cartoon depicts a young child responding to her teacher's enquiry 'Do you have any questions (about fractions)?' by asking 'Do you hate us?'.[1]
- Typically, mathematics educators see fractions as having multiple aspects, embedded in the complex conceptual field of multiplicative structures (Vergnaud, 2009). On the other hand, Hung-His Wu (1999) objects to this position on various grounds, appealing to the mathematician's dogma that mathematics is formal, abstract, simple, precisely defined. For example, the student is expected to 'understand a rational number as a point on the number line' (CCSSM, p. 43). How can

1 See Charles Schulz (November 7, 1991), *GoComics*, www.gocomics.com/peanuts/1991/11/07

> a number be a point? A full treatment of Wu's position would require a book (which I may yet write). Mathematicians tend to approach fractions in terms of computational properties and embeddedness in formal structures, such as groups, and to try to project that perspective onto learners.

CCSSM (pp. 6–8) has a very short section on 'Standards for mathematical practice' namely:

1. Make sense of problems and persevere in solving them
2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others
4. Model with mathematics
5. Use appropriate tools strategically
6. Attend to precision
7. Look for and make use of structure
8. Look for and express regularity in repeated reasoning

Together with a brief characterisation of each, with examples. As such, this is a fine list, but the teachers who plunge into CCSSM expecting enlightenment on how to cultivate such practices in their classrooms will find little. In particular, I find the treatment of Modelling (Standard 4) decidedly undernuanced and overly limited to a straightforward encoding of a described situation into a precise mathematical formulation.

Arguably, premature formalisation represents the most pervasive and harmful influence from mathematicians on mathematics-as-school-subject. For the sake of making an argument, let me list positions that can be found in the writings of mathematicians (sometimes bordering on caricature):

- The attitude that if you define everything with precision, build everything up logically, step by step, and they still don't get it, it's their fault.
- Mathematics-as-school-subject exists primarily for the preparation of the next cadre of research mathematicians.

Schools should teach a great deal of calculus, in particular, so that university mathematics instruction can hit the ground running.

- In lamenting what he perceives as a schism between mathematicians and mathematics educators, Michael Fried (2014, p. 4) expressed nostalgia for the time when 'asking about the distinction between mathematics and mathematics education would have been like asking about the distinction between mathematics and geometry'.
- In addressing the mathematical education of children, mathematicians tend to project their own images (or 'pictures', see Freudenthal, 1991, p. 131) of mathematics onto children.

Overall, it can be argued that mathematicians (of course, with many exceptions, have had harmful effects in multiple ways:

- Perpetuation of the Graecocentric narrative of the history of academic mathematics through a combination of laziness and ideology. In my opinion a line can be drawn between this and the manifestations of racism in contemporary classrooms.
- Denigration of the mathematical practices of those who make things that work (Chapter 17, this volume).
- A tendency to assume that being good at mathematics is not only a necessary but also a sufficient condition for being good at teaching mathematics.
- Alienation from, and perceived irrelevance of, mathematics combined with a propensity for intimidation.
- Failure to sew the seeds of criticality and agency in future citizens.
- Misuses of mathematics in the service of states.

As mentioned above, throughout history there have been exceptions. I will mention some personal heroes. Ubiratan D'Ambrosio brought to our field the necessary radically different kind of thinking that began to liberate Eurocentric anthropology and psychology from their imperial and colonialist roots. The influences of Hans Freudenthal and Jim Kaput on my thinking must be obvious in this chapter. (I vividly remember the latter commenting on a paper and drawing on the blackboard a large, amorphous creature, representing mathematics education. He then meticulously drew one toenail and commented 'We spend too much time analysing toenails on the creature when we should be analysing the creature'.) Reuben Hersh was one of the leaders in a radical reformulation of what philosophy of mathematics might be about, communicated accessibly what mathematicians do when they do mathematics, and illuminated the pervasive diversity within academic mathematics. Alan Schoenfeld, and his career, speak for themselves in his chapter in this volume.

Rethinking curriculum

Here I am using curriculum in the sense of a plan for the contents and sequencing of school mathematics. As a starting-point, remember the metaphor of inverting the pyramid, introduced above. Taking that position has heavy implications for content. Above all, combined with a shift in the centre of gravity from mere competence to understanding and problem solving, and attention to premature formalism, there could be a drastic reduction in the amount of 'technical' mathematics, including, as argued above, work with fractions and multiplying negative numbers (Hilton, 1984). Of course, there are protests against such a position. One such argument, that I find ill-founded, is that it hurts those children who are mathematically gifted. By way of counterarguments, I would characterise such giftedness as partly a cultural construct heavily loaded with connotations of cultural capital and that, in these days when masterclasses can be put online, enrichment is easily provided for those children who should (in whatever sense) have it (with careful provision to ensure such facilities are equitably accessible). As for students arriving at university with less technical knowledge under their belts, maybe the university teachers need to up their game. And, bearing in mind the adage that if you have four hours to chop wood, you should spend the first two hours sharpening the axe, if they arrive better able to 'think mathematically' (and enjoying mathematics rather than, at best, being rewarded by competence) that may be more than ample compensation.

As argued at various points, and see further below, in modern times mathematical modelling needs to be taken seriously, with a lot more attention to the socially and politically situated processes of assumption-making, simplification, mathematisation, interpretation, and communication of results. The historical alignment of mathematics with the physical sciences is discussed in Chapter 8, together with suggestions that this traditional alignment be reconsidered. The extension of mathematical modelling to social phenomena is reflected in the prominence of the use of mathematical techniques in social sciences such as experimental psychology (emphasis on measurement, psychometrics, and statistics).

Then there are what I think of as the rights of the child. As far as cognition goes, foremost of these is the right to sense-making. As far as identity goes, there are cultural rights, including access to an accurate (as far as possible) and balanced history of the development of academic mathematics, as well as an appreciation of the 'funds of knowledge', which is 'based on a simple premise: People are competent, they have knowledge, and their life experiences have given them that knowledge' (Gonzalez, Moll, & Amanti, 2005, p. ix). And then there are the multiple aspects of equitable treatment, educationally and personally.

Curriculum developers, in my opinion, can show a remarkable ability to fail to learn from history; this reflects, and partly explains, the stultifying inertia that characterises school mathematics, the slowness to embrace new content and resources. Lessons from the failure of the variety of attempts made under the banner of New Math have not been sufficiently absorbed. Paralleling trends in assessment, curriculum designers appear increasingly to benchmark against productions in other countries. This can lead to the error of importing a particular resource without the cultural embedding that makes it effective in its original milieu. It also encourages convergence (almost in a mathematical sense) with consequent implications for homogenisation.

If we look back as recently as the 1990s, at that time there were significant advances in assessment, even to the point of creating optimism (see review in De Corte, Greer, & Verschaffel, 1996, pp. 530–534). That has largely disappeared – by way of example, one only has to look at the fate of Smarter Balanced Assessment Consortium as narrated by Schoenfeld (Chapter 14, this volume). In a similar way, as described above, the potential of computers as reviewed by Kaput (1992) has yet to be adequately realised in classrooms.

And there are what might be called emotional rights. There is no reason why elementary mathematics instruction should not be an intellectual playground. There is no reason why mathematics teachers should tend to authoritarianism, but the subject certainly provides multiple opportunities for any such tendency. Mathematicians who love mathematics express sympathy for those who are alienated by it (or more accurately what they have been confronted with), but might spend more time thinking about whether they need to show intellectual empathy for the children who lack the facility with mathematics that they themselves typically enjoyed when young.

Teachers have rights, too, but that's another story.

Coherent long-term design

> Calculus might be regarded as a web of ideas that should be approached gradually, from elementary school onward, in a longitudinally coherent school mathematics curriculum. (Kaput, 1994, p. 78)

Kaput was talking specifically about calculus – which he suggests should be reconceptualised as 'the mathematics of change' (p. 152) and not necessarily built on the traditional foundation of algebra (pp. 77–78) – but the point applies equally to any major branch of mathematics. To give another example, instead of the framing 'the transition from arithmetic to algebra', the inherently algebraic nature of arithmetic may be recognised, and there are plenty of pedagogical moves to do just that.

Here I make what I see as vital points about curricular design being long-term, coherent, forward-looking, and mindful of conceptual obstacles and pedagogical dilemmas. At a very concrete level, a century ago, Edward Thorndike (1922) observed that children's mathematical conceptualisations are significantly framed by the examples to which they are exposed. A narrow range can result in a narrow understanding.

The lingering effects of behaviourism in folk pedagogy include a belief in the obviousness of the principle of monotonic and incremental movement along a simple/complex dimension, and the short-termism engendered by the desire to maximise scores on the next test. Efraim Fischbein pointed out the consequent dangers:

> From the educational point of view there is an important problem to be considered by curricula writers and by teachers. A certain interpretation of a concept or an operation may be initially very useful in the teaching process as a result of its intuitive qualities (concreteness, behavioral meaning etc.). But as a result of the primacy effect that first model may become so rigidly attached to the respective concept that it may become impossible to get rid of it later on. The initial model may become an obstacle which can hinder the passage to a higher-order interpretation – more general and more abstract – of the same concept. (Fischbein, 1987, p. 198)

A similar warning was expressed by De Morgan in relation to number and arithmetic:

> If we could at once take the most general view of numbers, and give the beginner the extended notions which he may afterwards attain, the mathematics would present comparatively few impediments. *But the constitution of our minds will not permit this.* (De Morgan, 1831/1910, p. 33, emphasis added)

Accordingly, it is essential to identify points at which conceptual change is difficult (and here history can be an indispensable guide) and then look for bridging resources. A clear example is the extension of multiplication and division beyond the positive integers to positive rational numbers, particularly those less than 1. The ramifications of this extension have been very extensively researched, in particular how 'multiplication makes bigger, division makes smaller' is no longer valid (here Thorndike's precept is particularly relevant). The consequent difficulties can be ameliorated, as discussed in Greer (1994), by pre-emptively including examples of multiplication and division by numbers less than 1 as early as possible and by the use of bridging example sets and representations. The point was well made by Cajori (1898):

> That, in the historical development, multiplication and division should have been considered primarily in connection with integers, is very natural. The same course must be adopted in teaching the young. First come the easy but restricted meanings of multiplication and division, applicable to whole numbers. *In due time, the successful teacher causes students to see the necessity of modifying and broadening the meanings assigned to the terms.* (p. 183, emphasis added)

As an overarching principle, the above considerations should be discussed with students. Fischbein, for example, has recommended telling students about historical examples showing that conceptual change is difficult for mathematicians too.

Many other examples come to mind of the consequences of lack of forethought. To a mathematician, it is obvious that when performing calculations on numbers that are measures of quantities, e.g., multiplying speed by time to get distance, the operation is invariant in relation to the numbers; for a child this principle is very far from evident, as shown by considerable research. Again, to a mathematician, multiplication is commutative, but in certain contexts it isn't, in the sense illustrated by the following example. To find the distance travelled by something at a constant speed of 0.75 miles per hour for 3 hours is instantly recognised as being found by multiplying the two numbers, but if it is 3 miles at 0.75 miles per hour, not so (many children will say the answer can be found by dividing 3 by 0.75, plausibly because they realise that the answer will be less than 3 and 'multiplication makes bigger, division makes smaller').

Modelling: From unreasonable effectiveness to reasonable ineffectiveness

The unreasonable effectiveness of mathematics. (Wigner, 1960, title)

The reasonable ineffectiveness of research on mathematics education. (Kilpatrick, 1981, title)

Eugene Wigner's seminal article addressed the question of how it is possible, for example, to predict through mathematics the movements of celestial bodies. Jeremy Kilpatrick, in relation to mathematics education, pointed out that the answer to 'How can we send a man to the moon, but cannot improve mathematics education?' is that the first is, however complex, a technical problem, while the second is a human problem. A similar contrast is evident in moving from the modelling of physical phenomena to the modelling of phenomena involving humans. Further, given the pervasiveness of what Skovsmose has characterised and analysed for decades as the formatting of our lives through mathematical modelling, it is important that the curriculum address the associated complexities.

The simple schematic for modelling in terms of mathematisation, development of implications, evaluation, and revision needs to be elaborated to include the following aspects:

- Considerations of who is doing the modelling and for what purpose; there is massive added complexity, particularly at the stages of mathematising the situation, including, in particular, what assumptions are made.
- Limitations imposed by technology and techniques available – these diminish with computing developments, but are still an issue for students.
- Evaluation of the outcomes of manipulating the model are also subject to the motivations of the modellers.
- Communication and dissemination of the results, especially some sense of their fragility (reasonable ineffectiveness); motivations of the modellers are also central at this point.

The foregoing considerations have massive implications for what should be taught – not just modelling in the sense of examples of how it is done, but questions of why. An extreme (in my view) counterargument was put, with admirable clarity, by André Toom (1999). His position was that, rather than viewing word problems as having anything to do with applications, the purpose of including such problems is simply to help teach pure mathematics and students to quickly learn the rules. The issue is pinpointed in the remarkable amount of discussion about the single equation $2 + 2 = 4$. As Houman Harouni (see Chapter 12, this volume) outlines, this discussion can become very rarified; I find the explanation by Reuben Hersh (1997, p. 16) straightforward and convincing, that the equation has a double meaning, as a statement of arithmetic, and as a description of what happens when 2 stable entities are put together, without interaction, with 2 other stable entities. To elucidate slightly, $2 + 2 = 4$ may afford a precise model – if I go to the bakery for donuts and my wife has said to get two for her, and two for myself, I could be in trouble if I come back with a number of donuts other than four. Or it could be totally wrong as a model – if your doctor says you can drink 2 pints a night and a second opinion confirms that recommendation, that is not a licence to drink 4 pints a night.

In Verschaffel, Greer, and De Corte (2000), we argued that what are called 'word problems' or, especially in the United States, 'story problems' could, indeed should, be treated as simple exercises in modelling. There is no reason why, through such problems, young children should not be introduced to the core insight that models may be exact, approximate, or plain wrong, and that it is possible to discriminate among those cases. Giyoo Hatano (1997) argued that the cost of increasing the demands on students by having them learn about complexities is too high; the position taken in this chapter that the cost of *not* doing so is also too high. In extension of this line of thinking, I would argue that mathematics education inculcates simplistic thinking. Children are taught the rules of the word problem game, foremost of which is that when you enter the mathematics classroom, you can ignore what you know about the real world and enter:

> A strange world in which you can tell the age of the captain by counting the animals on his ship, where runners do not get tired, and where water gets hotter when you add it to other water. (Back cover of Verschaffel, Greer, van Dooren, & Mukhopadhyay, 2009)

Early school mathematics can be seen as foundational in establishing not just the beginnings of understanding and competence, but also epistemological biases beneath the surface of mathematical content and techniques, including, in particular:

- The implicit assumption that essentially anything can be measured on a single dimension, and therefore individuals and groups can be measured in terms of that variable. The case of IQ provides a particularly clear and consequential example.
- The idea that real-world situations can be modelled unproblematically in terms of mathematical structures and operations and that once numbers and models have been specified, they cannot be disputed.

It does not have to be like that

Many issues about the development of mathematics-as-discipline by humankind were raised in Chapter 2. Likewise, there are fundamental questions in considering mathematics-as-school-subject:

- What are the relationships between the development of mathematics by humanity over millennia and the growth of mathematical understanding in an individual? How can a child be expected to come to grips with conceptual networks that took the combined intellectual resources of humankind millennia to create? How can this challenge be addressed within constructed environments?
- Why is school mathematics so alienating, and unused/unusable for so many (including highly intelligent people), problematic even for those who succeed, and loved by only a few (Hersh & John-Steiner, 2011)?
- Why do states/societies ask children to endure such stupidity?
- Why is systematic design, illuminated by study of the past, conspicuously lacking?

Above all, we should always return to the basic questions 'What is mathematics education *for*?'. Why could it not be different, and in what ways? Some thirty years ago, I was asked to say briefly what I had learned about mathematics education. I responded: 'For too many people, school mathematics is a personally and intellectually negative experience. It does not have to be like that'. That remains a good summary of how I feel.

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