NO PRICES NO GAMES!

FOUR ECONOMIC MODELS

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1 Equilibrium in the Jungle

The standard economic approach treats economic activity as voluntary: all involved parties are doing whatever they do of their own free will. When analyzed using the competitive equilibrium approach, economic agents operate within bounds set by a price system that they take as given, but their decisions are free — no one forces them to act. When analyzed using the game-theoretical approach, agents behave strategically, and in equilibrium they best respond to correct predictions about the other agents' behavior, and, again, no one can force anyone to take a particular action.

However, life is not just a series of voluntary actions. An agent might use power to seize assets from others or to force others to do things against their will. Resources are often transferred from one agent to another based on the exercise of power, rather than due to the satisfaction of mutual wants. While an agent can use power to force another to behave against his best interests, there is often no need to actually use power since the mere threat of doing so can be sufficient to persuade a weaker agent to give in.

Economic Theory typically ignores the use of power as a driver of social activity. In the words of Hirshleifer (1994) (see also Bowles and Gintis (1992) and Grossman (1995) who express similar sentiments):

> … the mainline Marshallian tradition has … almost entirely overlooked what I will call the dark side of the force — to wit, crime, war, and politics. … Appropriating, grabbing, confiscating what you want — and, on the flip side, defending, protecting, sequestering what you already have — that's economic activity too.

As the title of the book promises, we consider economic interactions that are harmonized without the emergence of a price system or the use of strategic

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deliberation. In this chapter, we shine a spotlight on the use of power by presenting a model where a power relation between the agents together with their preferences determine the outcome of their economic interactions.

The first notion of power that comes to mind is brute force. But power takes many softer forms. For example, power based on rank and seniority plays an important role in the allocation of resources, and the power of rhetoric or charm often allows one person to convince another to perform some action.

This chapter closely follows Piccione and Rubinstein (2007) which introduces and analyzes an elementary model of a society referred to as the *jungle*, in which economic transactions are governed only by coercion. The model consists of a set of agents with exogenous preferences over a set of assets and a power ordering of the agents. The ordering is unambiguous and known to all. Power means that a stronger agent is able to take things away from a weaker agent without the weaker agent's consent.

The jungle model is designed to mirror the standard model of an exchange economy. In both models, agents have preferences over assets and the total stock of assets is given. The distribution of power in the jungle replaces the initial distribution of assets in the market. Just as the acquisition of initial endowments is ignored in an exchange economy model, so is the acquisition of power ignored in the jungle model.

The jungle model makes no reference to property rights. An individual holds assets rather than owning them. There is no legal system that protects an individual's assets. Rather, a weaker agent can be forced to give up assets or coerced into an unfavourable exchange by a stronger agent. These features are in contrast to the standard exchange economy where property rights are perfectly enforced and exchanges are carried out only by mutual consent.

The solution concept we employ is called the *jungle equilibrium*. It is a feasible allocation of the assets such that no agent wishes to take assets from an agent (or agents) weaker than himself. In this chapter, we will apply versions of the concept to two different economies. The first is a version of Shapley and Scarf (1974)'s housing economy in which the set of assets is a discrete set of houses where each house can be occupied by only one agent and each agent can hold only one house. The second is the division economy where a bundle of divisible goods is allocated among the agents. Throughout, we will deal with standard issues, such as existence, uniqueness, and the two fundamental theorems of welfare.

1.1 The Housing Jungle: Model and Equilibrium

Recall that the housing economy is a tuple $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ where N is a finite set of n agents, X is a set of n houses, each agent $i \in N$ has a strict preference relation $\succsim^i$ over X (there are no indifferences), and F is the set of feasible profiles, which consists of all profiles $(x^i) \in X^N$ such that $x^i \neq x^j$ for every two agents i and j. This definition of F stipulates that every agent occupies exactly one house and every house is occupied by exactly one agent.

Note again that in this economy and throughout the book, there are no externalities in preferences (as in the standard market model). Each agent's preferences are defined over X, that is, an agent only cares about his own house and not about who occupies the others.

The jungle model's key ingredient is a power relation $\rhd$, which is a strict ordering (complete, asymmetric, and transitive) on the set of individuals. The term $i \rhd j$ is read as "agent i is stronger than agent j", which means that i can confiscate any house occupied by j. In summary,

Definition: Jungle Housing Economy

A **jungle housing economy** is a tuple $\langle N, X, (\succsim^i)_{i\in N}, F, \rhd\rangle$ where:

- $N = \{1, \ldots, n\}$ is the set of agents.
- X is the set of houses and $|X| = n$.
- $\succsim^i$ is agent i's strict ordering over the houses.
- F is the set of all profiles $(x^i)_{i\in N}$ such that $x^i \neq x^j$ for any two agents i, j.
- The *power relation* $\rhd$ is a strict ordering on the set of agents N. Without loss of generality, we assume that $1 \rhd 2 \rhd \cdots \rhd n$.

Comments on the notion of power

The power relation is exogenous: The model does not specify the source of power. As mentioned earlier, this is analogous to market settings where the initial endowments are taken as given without specifying their source. Naturally, one can think about models (not discussed here) in which the attainment of power (or initial endowments) is also a part of the model.

The exercise of power does not involve a loss of resources: We have in mind that an agent i who prefers a house currently occupied by a weaker agent j can confiscate it at no cost: i simply presents himself at j's door and j, recognizing his relative weakness, will move out. This is analogous to the standard exchange model where the exercise of trade and the enforcement of property rights are costless.

Power is exercised by an individual, not by a group: A stronger agent can force a weaker one to take an action, but a group of agents cannot form a coalition in order to force some action on another agent. Nor can the defendant then form a rival coalition. This is analogous to the standard exchange economy setting where agents act individually and coalitions cannot be formed (for example, for the purpose of price manipulation).

The power relation is transitive: One can think of situations in which it is not. For example, suppose that there are three components of power: agility, speed, and strength, and one agent can defeat another by being superior in a majority of them. A Condorcet-like configuration is possible where agent 1 is superior to agent 2 in agility and speed, agent 2 is superior to agent 3 in speed and strength, and agent 3 is superior to agent 1 in agility and strength. Thus, $1 \rhd 2 \rhd 3 \rhd 1$.

The outcome of a confrontation is deterministic: If i is stronger than j, then both are aware that, in any contest between them, i will win with certainty. However, uncertainty about the outcome of a confrontation is also plausible. One could think of a model where the power relation is replaced with a function that specifies for each pair of agents the probability of each of them winning a confrontation between them.

A jungle equilibrium is a profile of choices that is stable given the forces at play. In the jungle housing economy, there are two forces which can lead to instability. First, an agent prefers a house that is occupied by a weaker agent. Second, two individuals intend to occupy the same house. Formally:

Definition: Jungle Equilibrium

A **jungle equilibrium** for a jungle housing economy is a profile of houses (x^i) such that:

(i) There are no two agents $i, j \in N$ for which $i \rhd j$ and $x^j \succ^i x^i$.

(ii) The profile (x^i) is in F.

We start our investigation with an existence result:

Proposition 1.1: Existence of a Jungle Equilibrium

Every jungle housing economy has a jungle equilibrium.

Proof:

Let $\langle N, X, (\succsim^i)_{i \in N}, F, \rhd \rangle$ be a jungle housing economy. Recall that the agents are ordered by power, $1 \rhd 2 \rhd \cdots \rhd n$. Existence is shown using the *serial dictatorship* procedure: Agent 1 is assigned his favourite house $x^1 \in X$; agent 2 is assigned his favourite house from among the remaining houses, $x^2 \in X - \{x^1\}$; and successively, each agent, in order of power, is assigned his favourite house from among those remaining after houses have been assigned to all agents stronger than him. Since the number of houses equals the number of agents, the procedure assigns a house to every agent. Furthermore, the procedure assigns every house only once and thus the profile (x^i) is in F. The profile is a jungle equilibrium because for every i the house x^i is the $\succsim^i$-best among all the houses that are possessed by agents not stronger than him.

Note that the serial dictatorship procedure used in the proof is not the equilibrium concept itself but, rather, is a simple algorithm used to prove that an equilibrium exists. Proposition 1.1 above leaves open the possibility that other equilibria may exist. However, we will now show that, given the assumption that all preference relations are strict, the equilibrium is unique.

Proposition 1.2: Uniqueness

Every jungle housing economy has a unique equilibrium.

Proof:

Consider the jungle housing economy $\langle N, X, (\succsim^i)_{i\in N}, F, \rhd\rangle$. Assume, contrary to the claim, that (a^i) and (b^i) are two different equilibria of the jungle. Denote by i^* the strongest individual i for whom $a^i \neq b^i$. Suppose that $a^{i^*} \succ^{i^*} b^{i^*}$. Since the set of houses allocated to individuals 1 through $i^* - 1$ is the same in both (a^i) and (b^i), it must be that in (b^i), the house a^{i^*} is held by an agent j who is weaker than i^*. Thus, $i^* \rhd j$ and $b^j = a^{i^*} \succ^{i^*} b^{i^*}$ which contradicts (b^i) being an equilibrium.

1.2 The Jungle Equilibrium: Welfare

We move to discuss two fundamental welfare theorems. In abstract, the first states that, for any initial conditions, equilibrium outcomes are Pareto-optimal profiles; while the second states that Pareto-optimal profiles are equilibrium outcomes for some initial conditions. In the jungle housing economy, the initial condition is the power relation. We now bring proofs of the fundamental welfare theorems for the jungle housing economy (Abdulkadıroğlu and Sönmez (1998) show equivalent results that the set of allocations obtained by a serial dictatorship of some order is equal to the set of Pareto-optimal allocations).

Proposition 1.3: The First Welfare Theorem

The jungle equilibrium is Pareto optimal.

Proof:

Recall the assumption that preferences are strict. Let (x^i) be the jungle equilibrium. Assume that, contrary to the claim, there is a feasible profile (y^i) that Pareto dominates (x^i). Let i be the strongest agent for whom $y^i \neq x^i$. Then, $y^i \succ^i x^i$ and $x^j = y^i$ for some agent j weaker than i, contradicting the fact that (x^i) is a jungle equilibrium.

The above proof relies on the strictness of the agents' preferences. If some individuals have indifferences in their preferences, then a jungle equilibrium might not be Pareto optimal. For example, in the case of two agents and two houses a and b, if $a \sim^1 b$ and $a \succ^2 b$, then both $(x^1, x^2) = (a, b)$ and (b, a) are jungle equilibria, but the profile (a, b) is not Pareto optimal.

We now move to the second welfare theorem. Since the initial conditions for the jungle housing economy are a power relation, the appropriate second welfare theorem states that for any Pareto-optimal profile, there is a power relation for which the jungle equilibrium is precisely that profile. Recall that in the standard exchange model, the social planner assigns initial endowments to the agents with the expectation that trade between them will yield the desired allocation of the total endowment. Analogously, in the jungle housing economy, the social planner assigns the power relation with the expectation that the law of the jungle will yield the desired allocation of the houses.

Proposition 1.4: The Second Welfare Theorem

Given any housing economy $\langle N, X, (\succsim^i)_{i \in N}, F \rangle$ and Pareto-optimal profile (x^i), there exists a power relation $\rhd$ such that (x^i) is the unique jungle equilibrium of the jungle housing economy $\langle N, X, (\succsim^i)_{i \in N}, F, \rhd \rangle$.

Proof:

First, note that in every Pareto-optimal profile (x^i), at least one individual is allocated his favourite house: Otherwise, start with some agent i_0, and define i_{k+1} to be the agent who holds i_k's favourite house ($i_{k+1} \neq i_k$ because no agent's favourite house is his current house). Since N is finite, there will eventually be some l such that $k > l \geq 0$ and $i_{k+1} = i_l$. Then, assigning $y^{i_j} = x^{i_{j+1}}$ for each $l \leq j \leq k$, and keeping $y^j = x^j$ for all other agents, we obtain a feasible allocation (y^i) which Pareto-dominates (x^i).

We construct a power relation $\rhd$ as follows: Let i_1 be an agent for whom x^{i_1} is his first-best house and make him the most powerful agent. Now remove i_1 from the set of individuals and x^{i_1} from the set of houses. The inductive process continues as follows: at the beginning of the $k+1^{\text{st}}$ stage, k agents have been assigned power. The allocation of the remaining houses among the remaining agents is Pareto optimal; therefore, identify an agent i_{k+1} for whom $x^{i_{k+1}}$ is his favourite house from among $X - \{x^{i_1}, \ldots, x^{i_k}\}$ and make him the $(k+1)^{\text{st}}$-most powerful individual.

By construction, for each agent i, the house x^i is preferred by i over every house that is allocated to an individual weaker than him according to $\rhd$. Thus, (x^i) is a jungle equilibrium of $\langle N, X, (\succsim^i)_{i \in N}, F, \rhd \rangle$.

Externalities: To incorporate externalities, we modify the model by defining the agents' preferences over the set of feasible profiles (rather than the set of houses) and by allowing indifferences. The definition of a jungle equilibrium also needs to be modified. When deciding whether to confiscate a house, an agent compares the current profile to the one that would result if he does so. One way to proceed is by interpreting $i \rhd j$ to mean that agent i can force j to exchange houses: i takes over the house occupied by j *and* forces j to accept the house i previously occupied. Thus, an *equilibrium of the jungle with externalities* $\langle N, X, (\succsim^i)_{i \in N}, F, \rhd \rangle$ is a feasible profile (a^i) such that for no two agents $j, j' \in N$ is it the case that $j \rhd j'$ and $(b^i) \succ^j (a^i)$, where (b^i) is the allocation that differs from (a^i) only in the fact that $b^j = a^{j'}$ and $b^{j'} = a^j$.

In the model with externalities, a jungle equilibrium does not necessarily exist. For example, consider a case with 3 agents where $1 \rhd 2 \rhd 3$ and $X = \{a, b, c\}$. Think of the houses as being located clockwise on a circle: $a \to b \to c \to a$. Suppose that agent 1 top-ranks the three profiles where he is the clockwise neighbour of 2. Likewise, agent 2 top-ranks the three profiles where he is the clockwise neighbour of 1. There is no equilibrium because in any profile, agent 3 is the clockwise neighbour of either agent 1 or 2, in which case the other agent desires agent 3's position and is stronger than him. It is also easy to find an example with three individuals in which a jungle equilibrium exists but is not Pareto optimal.

1.3 Comparison to the Competitive Equilibrium

Shapley and Scarf (1974) used the extended housing economy for studying the notion of competitive equilibrium in a simple setting with discrete goods. Recall that the *extended housing economy* is a tuple $\langle N, X, (\succsim^i)_{i\in N}, F, (e^i)_{i\in N}\rangle$ where $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ is a housing economy and (e^i) is a feasible profile which is interpreted as an initial allocation of the houses. Thus, instead of a power relation, the housing economy model is enriched with the specification of an initial endowment for each agent. Shapley and Scarf (1974) define a *competitive equilibrium* for this extended economy to be a profile of prices (one real number to each house) and a profile of houses such that: (i) each agent prefers his assigned house to any that is not more expensive than his initial endowment and (ii) the housing assignment is feasible. Formally:

Definition: Competitive Equilibrium

A **competitive equilibrium** for an extended housing economy is a tuple $\langle (p_x)_{x\in X}, (x^i)_{i\in N}\rangle$ where $(p_x)_{x\in X}$ is a profile of prices and (x^i) is a profile of houses such that:

(i) For every individual i, the house x^i is $\succsim^i$-maximal in $\{x \mid p_{e^i} \geq p_x\}$.

(ii) The profile (x^i) is in F.

The following proposition, due to Shapley and Scarf (1974), shows that a competitive equilibrium exists. The proof, due to David Gale, uses an algorithm which is based on the notion of a top-trading cycle. Given any group of agents with initial endowments, a *top-trading cycle* is a cycle of agents all of whom most prefer the house of the next agent in the cycle from among those that the group members are endowed with. If an agent prefers his own house to all others then he makes a cycle of length one. We will see that a top-trading cycle always exists.

The top-trading cycle algorithm proceeds as follows: at each stage, a top-trading cycle is identified. Each agent in the cycle is exclusively assigned the house of the next agent in the cycle (which he prefers from among the houses that were not assigned previously). All houses in the cycle are assigned the same price, which is lower than the prices of all previously assigned houses, and both the assigned agents and the assigned houses are removed.

Proposition 1.5: Existence of Competitive Equilibrium

For any extended housing economy, a competitive equilibrium exists.

Proof:

Let $\langle N, X, (\succsim^i)_{i\in N}, F, (e^i)_{i\in N}\rangle$ be an extended housing economy. We first show that a top-trading cycle exists for every group of agents G. Start arbitrarily with an agent $i_0 \in G$, and define $i_{k+1} \in G$ as the initial holder of i_k's favourite house from the set of houses belonging to G. Since the group is finite, there will eventually be some l such that $k \geq l \geq 0$ and $i_{k+1} = i_l$. Then, the sequence $(i_l, \ldots, i_k)$ constitutes a top-trading cycle. See Figure 1.1 for an illustration of the argument where $l = 2$ and $k = 5$.

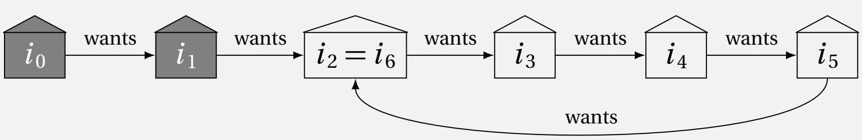

Figure 1.1 The Top–Trading Cycle algorithm.

The algorithm constructs a partition $\{I_1,\ldots,I_l,\ldots,I_L\}$ of N as follows: First, find a top-trading cycle from the group of all agents. Set I_1 to be the set of members of this cycle and assign to each of them the house he most prefers. Continue inductively: at stage $l+1$, find a top-trading cycle from among the group $N-I_1-\ldots-I_l$ and for each member of the cycle assign the house which he most prefers from among those initially held by the group. Set I_{l+1} to be the set of members in the cycle. Continue in this fashion until a partition is completed. Choose a sequence of numbers $p_1 > p_2 > \ldots > p_L > 0$ and, for each $x \in X$, define $p_x = p_l$ where the agent who initially occupies x is in I_l. The assigned profile (x^i) together with the price vector (p_x) constitutes a competitive equilibrium because $(x^i) \in F$, and every agent i in I_l chooses his favourite house from within his "budget set", namely the set of houses initially held by the members of $I_l \cup \ldots \cup I_L$.

Comparing the above construction to that of the jungle equilibrium clarifies the source of power in the market vs. the source of power in the jungle. In Gale's construction, in each round some agents obtain their favourite house from among those not allocated in previous rounds. So too in the jungle equilibrium. However, in the case of competitive equilibrium, the order is determined by the existence of a "top-trading cycle" which indicates the parties' joint interest in making an exchange, whereas in the jungle the order is determined by power, independently of the agents' preferences.

Given that the preference relations are assumed to be strict, there is a unique competitive equilibrium allocation (for a proof, see Osborne and Rubinstein (2023)). However, this allocation can be supported by many price systems, and it can even be that one house is more expensive than another in one equilibrium price system but less expensive in another.

The two fundamental welfare theorems hold for the competitive equilibrium in this model:

(a) Any competitive equilibrium $\langle(p_x),(x^i)\rangle$ is Pareto-optimal: if $(y^i)\in F$ Pareto dominates (x^i) then $p_{y^i}\geq p_{x^i}$ for all i with strict inequality for any agent i for whom $y^i\succ^i x^i$ and thus $\Sigma_{i\in N}\, p_{y^i}>\Sigma_{i\in N}\, p_{x^i}$ although the two sums are equal.
(b) For any Pareto-optimal allocation (x^i) there is a price vector (p_x) such that $\langle(p_x),(x^i)\rangle$ is a competitive equilibrium. By Proposition 1.5 a competitive equilibrium exists for the extended economy with the initial allocation (x^i). Its allocation (y^i) is weakly Pareto superior to (x^i) and since (x^i) is Pareto-optimal it must coincide with (x^i). Therefore, if we start with $(e^i)=(x^i)$ the proof constructs a competitive equilibrium in which each agent i keeps x^i.

Power and Wealth: Since the jungle equilibrium is Pareto optimal, it can be supported by prices as a competitive equilibrium. This invites a natural question: what is the relationship between power and wealth?

First, there is always a price system in which "stronger" in the jungle economy means "richer" in the competitive equilibrium of the extended housing economy with the initial endowment profile being the jungle equilibrium of the jungle economy. Formally, let (x^i) be the jungle equilibrium in the housing economy jungle $\langle N,X,(\succsim^i)_{i\in N},F,\rhd\rangle$. The extended housing economy $\langle N,X,(\succsim^i)_{i\in N},F,(e^i=x^i)_{i\in N}\rangle$ has a competitive equilibrium $\langle(p_x),(x^i)\rangle$ where $p_{x^i}>p_{x^j}$ whenever $i\rhd j$. However, other equilibrium price vectors may exist. For example, if the strongest agent top-ranks his own house while all other agents bottom-rank it, then there also exists a competitive price vector in which the strongest agent is the poorest.

In fact, if we modify the economy somewhat, then there may be no jungle equilibrium in which the statement "stronger = richer" holds. For example, recall the clubs economy where each agent chooses one club from the set X, and no more than q_x agents can choose club x. Consider the economy with 4 agents, where $X=\{a,b\}$ and $q_a=q_b=2$. If the preferences are such that agent 1 prefers a and all other agents prefer b, then the unique jungle equilibrium is (a,b,b,a). However, in this equilibrium, every agent obtains his first-best club except for agent 4 and to prevent agent 4 from getting what he wants it must be that $p_b>p_a$. Thus, any price vector which supports the jungle equilibrium allocation must have the property that the strongest agent is the poorest.

1.4 Comments on the Jungle Equilibrium

Comparative statics: The jungle equilibrium satisfies the expected comparative statics property that advancing an agent in the power ranking cannot hurt the agent. To see this, recall that there is a unique jungle equilibrium and it can be calculated via a serial dictatorship procedure. When an individual agent becomes stronger, all agents who are still stronger than him will continue to make the same choices, while the individual now gets to choose earlier and, therefore, has a strictly larger set of houses to choose from.

On the other hand, in the case of competitive equilibrium, improving an agent's initial house endowment, according to his own preferences, might make him worse off in equilibrium. Although the new house is better for him, it might be unattractive to other agents. Thus, when applying the top-trading cycle algorithm, it could be that he initially appeared in the first cycle and, after the "improvement", he now appears in the last cycle and, therefore, ends up worse off in the new equilibrium than in the old one.

Manipulability: The jungle equilibrium is immune to preference misrepresentations by an agent. Again, the unique jungle equilibrium can be calculated by the serial dictatorship algorithm. When it is an agent's turn to choose, the set of alternatives that he chooses from is unaffected by his declared preferences, and, thus, he can do no better by misrepresenting his preferences. This non-manipulability property also holds for competitive equilibria.

Indifferences: Even if some of the agents' preferences are not strict, the serial dictatorship procedure still produces a jungle equilibrium. However, it is not necessarily unique since, when an agent has to make a choice, he might have more than one maximal option and each produces a different equilibrium. Note that indifferences can also create a multiplicity of competitive equilibrium profiles in the housing economy market.

Equilibrium and Dynamics: The jungle equilibrium concept is static, like most solution concepts in Economic Theory. The following is an example of dynamics that lead to a jungle equilibrium: At the beginning, all agents are assigned to be "*homeless*". At stage $t+1$, given the assignment of the agents at stage t to $X \cup \{homeless\}$, every homeless agent chooses his favourite house from among those that, at the end of stage t, are either: i) vacant or ii) assigned to an agent weaker than him. Every agent who currently occupies a house chooses to stay there. At the end of stage $t+1$, if a house is chosen by only one agent, then he settles there. If more than one agent chooses the same house, then the strongest among them settles there and all the rest remain *homeless*.

Proposition 1.6: Equilibrium Dynamics

The above dynamics converges in at most n stages to the jungle equilibrium.

Proof:

Let H_t be the set of homeless agents at the beginning of stage t and i_t be the most powerful among them. If there are any homeless agents at stage $t+1$, then $i_t \rhd i_{t+1}$: To see why, note that at stage t, i_t will obtain a home because all homeless agents are weaker than him and so he will win at any home which he approaches. Furthermore, all agents stronger than i_t remain in their homes as no one challenges them. Thus, in the beginning of stage $t+1$, all homeless agents must be weaker than i_t. Therefore, after at most n stages, all agents have a home and the process terminates at a profile (x^i).

Suppose that (x^i) is different than the jungle equilibrium profile (y^i). Take i to be the strongest agent for whom $x^i \neq y^i$. Thus, $i \rhd j$ where j is the agent who holds y^i, i.e. $x^j = y^i$.

By Proposition 1.2, y^i is $\succ^i$-maximum in $X - \{y^1, \dots, y^{i-1}\} = X - \{x^1, \dots, x^{i-1}\}$ and therefore $y^i \succ^i x^i$. At the stage in the algorithm where i selected x^i it must be that y^i was being held by someone stronger than i. But, in the algorithm, when a house changes hands, it can only go to someone stronger so as it eventually reaches j it must be that $j \rhd i$, a contradiction.

A different power relation for each house: A key assumption in the jungle model is the uniformity of the power relation: if an agent i is able to evict agent j from one house, then he is able to evict him from any house. An extension of the model allows for dependence of the power relation on the house in dispute. Suppose that, for each house $x \in X$, there is a strict power ordering $\rhd_x$ where $i \rhd_x j$ means that agent i is stronger than agent j in a fight over house x. That is, if agent j occupies x and $i \rhd_x j$, then agent i can confiscate x. An equilibrium in the economy with house-dependent power relations $\langle N, X, (\succsim^i)_{i\in N}, F, (\rhd_x)_{x\in X}\rangle$ is a profile (x^i) such that there are no two agents i and j such that i prefers the house occupied by j to the house he occupies ($x^j \succ^i x^i$) and i is stronger than j regarding x^j ($i \rhd_{x^j} j$).

As commented on in Rubinstein and Yıldız (2022), the notion of a jungle equilibrium in $\langle N, X, (\succsim^i)_{i\in N}, F, (\rhd_x)_{x\in X}\rangle$ is equivalent to pairwise stability in an auxiliary two-sided matching problem between N and X where each agent $i \in N$ has the preference $\succsim^i$ over X and each house $x \in X$ has the preference relation $\rhd_x$ over N. A profile (x^i) is pairwise stable if there is no pair i and x^j such that i prefers x^j over x^i ($x^j \succ^i x^i$) and x^j "prefers" i over j ($i \rhd_{x^j} j$). Therefore, a profile is pairwise stable in the auxiliary matching problem if and only if it is a jungle equilibrium with house-dependent power relations.

Gale and Shapley (1962) showed, using the deferred acceptance algorithm, that a pairwise stable matching exists in any two-sided matching problem. Thus, in the jungle with house-dependent power relations, a jungle equilibrium also exists. Since the pairwise stable matching need not be unique, neither is the jungle equilibrium when the power relation is house-dependent. Finally, Gale and Sotomayor (1985)'s analysis implies that there is always a jungle equilibrium (x^i) which is *weakly* Pareto optimal, in the sense that there is no assignment (z^i) such that $z^i \succ^i x^i$ for every $i \in N$.

1.5 The Division Jungle

We now apply the jungle concept to a version of the division economy. To the definition of a division economy from Chapter 0, we add a profile $(X^i)_{i\in N}$ of personal consumption sets, which represent bounds on each agent's ability to consume. These sets can be thought of as either physical limits on what a person can consume or what possessions he can protect. Note that, in the housing economy, there is an implicit assumption of a similar nature, namely that an agent can hold only one house. The following is the formal definition of a jungle division economy (throughout, when comparing bundles, the notation $x \le y$ means that $x_k \le y_k$ for every commodity k):

Definition: Jungle Division Economy

A **jungle division economy** is a tuple $\langle N, (X^i)_{i\in N}, (\succsim^i)_{i\in N}, F, \rhd\rangle$ where:

- $N = \{1, \ldots, n\}$ is the set of agents.
- $X^i \subseteq \mathbb{R}^K_+$ is agent i's personal consumption set in a K-commodity world. The sets X^i are assumed to be compact, convex, and satisfy free disposal (that is, if $x^i \in X^i$, $y \in \mathbb{R}^K_+$ and $y \le x^i$, then $y \in X^i$).
- $\succsim^i$ are preferences over X^i and assumed to satisfy continuity, strict monotonicity, and strict convexity.
- F is the set of all profiles of bundles (x^i) such that:
  (i) $x^i \in X^i$ for all i, and
  (ii) $\Sigma_{i\in N} x^i \le e$ where $e \in \mathbb{R}^K_+$ is an aggregate bundle available for distribution among the agents.
- $\rhd$ is a strict power ordering over N.

Given a profile (x^i), denote the "leftover" bundle $e - \Sigma_{i\in N} x^i$ as x^0.

We now turn to modifying the definition of a *jungle equilibrium* to fit the division jungle. There are (at least) two possible definitions that coincide with that of the housing economy. The first is a *strong jungle equilibrium* which is a feasible profile such that no agent can assemble a preferable bundle by combining his own bundle with *all* bundles held by weaker agents and the leftover bundle. By this definition, the stability of a profile is disturbed by the possibility that an agent can attack more than one weaker agent. The second definition is a *weak jungle equilibrium*, which is a feasible profile such that no agent can assemble a preferable bundle by combining his own bundle with *one* other that is either held by a weaker agent or is the leftover bundle. Formally:

Definition: Strong Jungle Equilibrium

A **strong jungle equilibrium** is a feasible profile (x^i) with the property that there is no agent i and bundle $y^i \in X^i$ such that:
(i) $y^i \succ^i x^i$.
(ii) $y^i \leq x^i + \Sigma_{i \triangleright j} x^j + x^0$ (the agent takes from weaker agents and from the leftover bundle and potentially disposes of some of his possessions).

Definition: Weak Jungle Equilibrium

A **weak jungle equilibrium** is a feasible profile (x^i) with the property that there is no agent i and bundle $y^i \in X^i$ such that:
(i) $y^i \succ^i x^i$.
(ii) Either (a) or (b) holds.
(a) $y^i \leq x^i + x^j$ for some j for whom $i \triangleright j$ (the agent steals from a single weaker agent and then may dispose of some of his possessions); or
(b) $y^i \leq x^i + x^0$ (the agent takes from the leftover bundle and then may dispose of some of his possessions).

Note that the above definitions use inequalities rather than equalities. This is because, when a stronger agent seizes other resources, he might be put outside of his consumption set and, thus, needs either to take less or to dispose of some goods in order to remain in his consumption set. Obviously, any strong jungle equilibrium is also a weak jungle equilibrium.

Proposition 1.7: Strong Jungle Equilibrium: Existence and Uniqueness

There exists a unique strong jungle equilibrium.

Proof:

Here again, we proceed by applying the serial dictatorship procedure and constructing a feasible profile (x^i) as follows: Start with the strongest agent 1 and define x^1 as the $\succsim^1$-best bundle in the set $\{z \in X^1 \mid z \leq e\}$ which is closed and convex. Proceed inductively by defining x^i to be the $\succsim^i$-best bundle in the closed and convex set $\{z \in X^i \mid z \leq e - \Sigma_{j=1}^{i-1} x^j\}$. The profile (x^i), which is feasible by construction, is a strong jungle equilibrium. By the same proof as that of Proposition 1.2, the equilibrium is unique.

Monotonicity is not used in the above proof. For existence, only continuity of the preference relations is needed to ensure that the inductive procedure used in the proof leads to a strong jungle equilibrium. Strict convexity guarantees uniqueness.

Example: The Pie Jungle

In the **pie jungle**, there is a single pie of size 1. The consumption set for every agent i is $X^i = X = [0,1]$ where $x \in X$ is interpreted as taking x of the pie. We deviate slightly from the assumptions made above and assume that every agent i has strictly convex preferences on X with a peak at $peak^i$ (and thus the preferences are not monotonic). In this case, the weak and strong jungle equilibria coincide, and the unique strong jungle equilibrium is as follows: Let m be the minimal number for which $\Sigma_{i=1,\ldots,m} peak^i > 1$. Each of the $m-1$ strongest agents chooses his own peak, the m^{th} agent gets the leftovers and the rest get nothing. If there is no such m, that is, if $\Sigma_{i=1,\ldots,n} peak^i \leq 1$, then all agents choose their respective peaks.

Is there a weak jungle equilibrium that is not a strong jungle equilibrium? Yes. Consider the division economy with three agents and three commodities, an aggregate endowment $(1,1,1)$, and for each agent i his consumption set is $X^i = \{(x_1, x_2, x_3) \mid \sum x_k \leq 1\}$ and his preferences are represented by the utility function $u^i(x_1,x_2,x_3) = x_1 + x_2 + x_3 + x_1 x_2 x_3$. The power relation $1 \rhd 2 \rhd 3$ and the profile $x^1 = (1,0,0)$, $x^2 = (0,1,0)$, and $x^3 = (0,0,1)$ is a weak jungle equilibrium since an agent cannot improve himself by attacking *one* other agent. But, it is not a strong jungle equilibrium because agent 1 can improve himself by attacking the two weaker agents and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \succ^1 (1,0,0)$. The strong equilibria are when all agents get $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ together with any power relation.

The following proposition (whose proof is more technical than the others in the book) states that the two definitions are equivalent under "smoothness" assumptions on both the agents' preferences and on the consumption sets. Essentially, smoothness means that indifference curves and the frontiers of the consumption sets are smooth, in the sense that there is a unique tangent at any point. Formally, we will use the term *smooth* to describe an economy if the following holds for every i:

- Agent i's preference relation is represented by a strictly quasiconcave, increasing and continuously differentiable utility function $u^i : \mathbb{R}^K_+ \to \mathbb{R}$, with a strictly positive gradient vector $\nabla u^i(x)$ at every bundle x.
- There exists a strictly quasiconvex and continuously differentiable function $g^i : \mathbb{R}^K_+ \to \mathbb{R}$ such that $X^i = \{x^i \in \mathbb{R}^K_+ \mid g^i(x^i) \leq 0\}$ with a gradient $\nabla g^i(x)$ that is a strictly positive vector at every bundle x.

Proposition 1.8: Equivalence of the Jungle Equilibrium Definitions

For a smooth jungle economy, the two definitions of jungle equilibrium coincide.

Proof:

Let (x^i) be a weak jungle equilibrium which is different than the strong jungle equilibrium (y^i). We can suppose that $x^1 \neq y^1$, since otherwise induction could be applied to the jungle economy with players $N \backslash \{1\}$ and an adjusted endowment vector $e - x^1$.

The bundle y^1 is the unique $\succ^1$-maximum bundle in $\{x \in X^1 \mid x \leq e\}$. Therefore, $y^1 \succ^1 x^1$ and in turn $e_k \geq y^1_k > x^1_k$ for some k. Thus, $x^1 \neq e$. Also, it must be that $g^1(x^1) = 0$. If not, that is $g^1(x^1) < 0$, then since $e_k > x^1_k$, agent 1 could improve upon x^1 by seizing a small amount of good k either from an agent who holds some of that good (recall that agent 1 is the strongest agent) or from the leftovers, contradicting that (x^i) is a weak jungle equilibrium.

By definition, $u^1(y^1) > u^1(x^1)$ and $g^1(y^1) \leq g^1(x^1) = 0$. Perturb y^1 by removing a small amount of good k (for which $y^1_k > x^1_k$) so that, by continuity, $u^1(y^1) > u^1(x^1)$ and $g^1(y^1) < g^1(x^1)$. Then, by the assumptions that u^1 is strictly quasiconcave and that $\nabla u^1(x^1) \neq 0$, it follows that $\nabla u^1(x^1) \cdot \big(y^1 - x^1\big) > 0$. Likewise, it follows that $\nabla g^1(x^1) \cdot \big(y^1 - x^1\big) < 0$. By the Lemma below, there is a vector z in $\mathbb{R}^K$ such that (i) $z_k > 0$ for some k for which $y^1_k - x^1_k > 0$; (ii) $z_l < 0$ for some l for which $y^1_l - x^1_l < 0$; (iii) $z_h = 0$ for all $h \neq k, l$; and (iv) $\nabla u^1(x^1) \cdot z > 0$ and $\nabla g^1(z^1) \cdot z < 0$. That is, z is a vector which is non-zero in only two components, k and l. With respect to k, the component z_k is positive, and the fact that $y^1_k > x^1_k$ means there is an $i \neq 1$ (which can be 0) such that $x^i_k > 0$. With respect to l, the component z_l is negative, and the fact that $y^1_l < x^1_l$ means that $x^1_l > 0$. By (iv), for small enough $\varepsilon > 0$, adding εz_k units of commodity k and subtracting $-\varepsilon z_l$ units of commodity l strictly improves agent 1's utility and keeps him within his consumption set. Let ε be small enough so that εz_k units of commodity k can be taken from one weaker agent (or from the leftover x^0). This contradicts (x^i) being a weak equilibrium.

Lemma:

Let a and b be strictly positive vectors in $\mathbb{R}^n$, and suppose that $a \cdot z > 0$ and $b \cdot z < 0$ for some $z \in \mathbb{R}^n$. Then, there exists $\Delta \in \mathbb{R}^n$ such that: (i) $\Delta_k > 0$ for some k for which $z_k > 0$; (ii) $\Delta_l < 0$ for some l for which $z_l < 0$; (iii) $\Delta_h = 0$ for all $h \neq k, l$; and (iv) $a \cdot \Delta > 0$ and $b \cdot \Delta < 0$.

Proof:

First note that there are l and k such that $z_l < 0$ and $z_k > 0$. Define $\lambda_m = \frac{a_m}{b_m}$ for every m. Let λ_h be the smallest λ_l associated with $z_l < 0$. It is impossible that $\lambda_h \geq \lambda_k$ for all k such that $z_k > 0$, since in that case $a \cdot z = \Sigma_{i=1}^n \lambda_i z_i b_i \leq \Sigma_{i=1}^n \lambda_h z_i b_i = \lambda_h b \cdot z$, contradicting $b \cdot z$ being negative and $a \cdot z$ being positive. Thus, there is k such that $\lambda_h < \lambda_k$ and $z_k > 0$. Any vector $\Delta = (0, \ldots, 0, \Delta_k, 0, \ldots, 0, \Delta_h, 0, \ldots, 0)$ satisfying $\Delta_k > 0$, $\Delta_h < 0$ and $\frac{b_k}{b_h} < \frac{-\Delta_h}{\Delta_k} < \frac{a_k}{a_h}$. Thus, $a \cdot \Delta > 0$ and $b \cdot \Delta < 0$.

1.6 The Division Jungle: Comments on Welfare

The first fundamental welfare theorem: For the division economy, a first fundamental welfare theorem still holds, namely, the strong jungle equilibrium is Pareto optimal. The proof is similar to that of the housing economy: Suppose (z^i) is a strong jungle equilibrium which is not Pareto optimal. Let $(y^i) \in F$ be a Pareto-superior profile. Let j be the strongest agent for whom $y^j \neq z^j$. Both y^j and z^j belong to the set $\{x^j \in X^j \mid x^j \leq e - \Sigma_{i=1}^{j-1} y^i\} = \{x^j \in X^j \mid x^j \leq e - \Sigma_{i=1}^{j-1} z^i\}$. Since z^j is j's unique top-ranked bundle in this set, it holds that $z^j \succ^j y^j$, a contradiction. Note that for non-smooth economies a weak jungle equilibrium might be not Pareto optimal.

The second fundamental welfare theorem: On the other hand, the second fundamental welfare theorem cannot have an analogue for the division jungle. This is because in the division jungle there are finitely many power relations and typically infinitely many Pareto-optimal profiles. This cardinality mismatch implies that for a division economy, not every Pareto-optimal profile can be obtained as a jungle equilibrium by appending some power relation to that economy.

Power and wealth: Making a statement about the relation between power and wealth in the division economy is more involved than in the housing economy since the existence of a competitive equilibrium price vector that supports the jungle equilibrium is not guaranteed, even if all of the agents' consumption sets are the same. For a discussion of this issue, see Piccione and Rubinstein (2007).

1.7 A Didactic Perspective

The discussion in this chapter also has a didactic purpose, as expressed in the personal concluding remarks made by one of us in Piccione and Rubinstein (2007), which are essentially quoted here (with some small changes):

> When I present the model in public lectures, I ask the audience to imagine that they are attending the first lecture of a course at the University of the Jungle, entitled Introduction to the Principles of Economics. The analogy of such a presentation to the way we introduce the market equilibrium in a standard Microeconomics course serves as a device to shed light on the implicit message that Microeconomics students receive from us.
>
> Being faithful to the classical economic tradition, the jungle model does not stray far from the standard exchange economy. We use terminology that is familiar to any economics student. After having defined the notion of jungle equilibrium, we conduct the same type of analysis that can be found in any microeconomics textbook on competitive equilibrium. We show existence and then discuss the first and second fundamental welfare theorems. We emphasise the analogy between the initial endowments in an exchange economy and the initial distribution of power in the jungle: both are used to determine

the equilibrium distribution of commodities among the agents. Were I teaching this model, I would also add the standard comments regarding externalities and the place for government intervention.

There are arguments which attempt to dismiss the comparison between markets and jungles:

One might argue that the market has the virtue of providing incentives to "produce" and to enlarge the size of the "pie" to be distributed among the agents. On the other hand, one could also argue that the jungle provides incentives to develop power. In the market economy, agents invest effort in producing more goods. In the jungle economy, agents invest effort in becoming stronger, an asset for a society that needs to defend itself against invaders or invade others in order to accumulate resources.

One might argue that market mechanisms preserve resources that would otherwise have been wasted in conflict. Note, however, that under complete information a stronger agent can persuade a weaker one to part with his goods using only the threat of force. Societies often create rituals that help individuals gauge the power of others and thereby avoid the costs of conflict. Under incomplete information, the market also wastes resources. And finally, I have not mentioned the obvious transaction costs that are also associated with market institutions.

One might argue that labour is a good that should be treated differently. However, the long history of slavery shows this to be inaccurate.

One might also argue that the virtue of the market system is that it exploits people's natural desire to acquire wealth. In contrast, the jungle just uses people's natural willingness to exercise power and to dominate.

Obviously, I am not arguing in favour of adopting the jungle system. The comparison between the jungle and market mechanisms depends on our assessment of the characteristics with which agents enter the model. If the distribution of the initial holdings in the market reflects social values that we wish to promote, we might regard the market outcome as agreeable. However, if the initial wealth is allocated unfairly, dishonestly or arbitrarily, then we might not favour the market system. Similarly, if power is desirable then we might advocate for the jungle system, but if the distribution of power reflects brute force that threatens lives then we would clearly not be in favour.