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FOUR ECONOMIC MODELS

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2 The Permissible and the Forbidden

Picture in your mind a family consisting of n members. The grandparents have prepared a holiday feast and all are sitting happily around a long table. When the main dish is served, the grandparents act as dictators, putting a portion of it on each family member's plate and making sure they eat it to the last bite. And then, dessert arrives and with it a dramatic turn of events. Grandma and Grandpa enter the room with their famous homemade pie. Everyone loves their pie and gazes eagerly at its entrance. Given the chance, each family member would gladly eat more than $1/n$ of the pie. At this point, the grandparents declare that they will not interfere in the division of the pie and will let the younger generation use their academic knowledge to decide how the pie is divided.

One member of the family, an economist, suggests that each family member should be endowed with $1/n$ of the pie and — since some perhaps appreciate the pie more, while others perhaps less — a market should operate under the table where members can exchange slices of the pie for money. Another member of the family, a game theorist, suggests that the grandparents conduct an auction. He claims that this might be fun and, more importantly, the pie will be divided optimally. Hopefully, in your family, neither markets nor auctions are used to resolve such a conflict and, instead, harmony is achieved by means of a social norm: each family member does not dare to even consider taking more than the socially acceptable amount, say q, of the pie.

Obviously, not every q will bring harmony to the family. If $q > 1/n$, then a family crisis would erupt since there would not be enough pie to satisfy the family members. All family members would race to get their slice, and some will be disappointed because they are unable to realize their anticipation of eating q of the pie. If $q < 1/n$, then no conflict arises, but the members of the family would feel uneasy looking at the leftovers on the table and, next year,

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would feel justified in taking a bit more. If $q = 1/n$, then harmony prevails. It is optimal for each family member to take q, and any loosening of the norm will lead to demands which cannot be satisfied.

We think of a bound on the portion that one can take as an example of a natural social norm that specifies what is considered permissible ("done") and forbidden ("not done"). Such a norm resolves the family's allocation problem but not with prices or games.

Following Richter and Rubinstein (2020), we analyze the Y-equilibrium concept. It is defined as a set of permissible alternatives (which is the same for all agents) combined with a profile of choices (one for each agent) such that:

(i) each agent's choice is optimal from among the permissible alternatives;
(ii) the profile of choices is feasible; and
(iii) the set of permissible alternatives is maximal in the sense that there is no superset of permissible alternatives from which a profile satisfying (i) and (ii) can be found.

By this definition, two forces make a permissible set unstable: the first modifies the permissible set in the case that the profile of (intended) choices is not feasible, while the second loosens restrictions on the permissible set as long as a new profile of optimal choices is feasible.

The Y-equilibrium concept reflects a decentralized institution for achieving harmony in a society. We envision that, without a central authority, the same invisible hand that calculates equilibrium prices so "effectively" is also able to determine a maximal set of permissible alternatives that are compatible with self-maximizing behavior. The above forces adjust the social norm until harmony is achieved. While we do not provide a general dynamic process that converges to Y-equilibrium, in Richter and Rubinstein (2020), for several examples, we demonstrated natural *tâtonnement*-like processes that lead to a Y-equilibrium.

In standard economic settings, equilibrium prices can also be thought of as being determined by a central authority (or platform) that wishes to ensure that trade is viable. Similarly, one can think of the permissible set in a Y-equilibrium as a norm dictated by an authority that wishes to achieve harmony in society without imposing any unnecessary restrictions on the individuals.

We now proceed to the formal definition of the equilibrium notion.

2.1 The Y-Equilibrium Concept

Recall that an *economy* is a tuple $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ where N is the set of agents, X is the set of alternatives that each agent chooses from, $\succsim^i$ is agent i's preferences on X, and $F \subseteq X^N$ is the set of feasible choice profiles.

A candidate for an equilibrium is a *configuration* which consists of a subset of X, called a permissible set, together with a profile of choices:

Definition: Configuration

A **configuration** is a pair $\langle Y, (y^i)_{i\in N}\rangle$ where $Y \subseteq X$ and $(y^i)_{i\in N}$ is a profile of elements in Y. We refer to Y as a **permissible set** and to $(y^i)_{i\in N}$ as an **outcome**.

As explained in Chapter 0, a candidate for a solution in this book has a structure analogous to that of a competitive equilibrium. It is comprised of a profile of choices (one for each agent) and an additional parameter. In a configuration, the additional parameter is a permissible set, that is taken by all agents as given and uniformly binds the choices of all agents. Analogously, in a competitive equilibrium, the additional parameter is a price system, that is taken by all agents as given and uniformly binds the exchanges of all agents.

Before defining the equilibrium concept, we need an additional concept: a para-equilibrium is a configuration where each individual maximizes his interests given the permissible set and the resulting choice profile is feasible.

Definition: Para-equilibrium

A **para-equilibrium** is a configuration $\langle Y,(y^i)\rangle$ satisfying:

(i) For all i, y^i is a $\succsim^i$-maximal alternative in Y.

(ii) The profile (y^i) is in F.

A Y-equilibrium is a para-equilibrium such that any expansion of the permissible set will lead to a violation of feasibility if agents self-maximize with respect to the expanded permissible set.

Definition: Y-equilibrium

A **Y-equilibrium** is a para-equilibrium $\langle Y,(y^i)\rangle$ such that there is no para-equilibrium $\langle Z,(z^i)\rangle$ for which Z is a strict superset of Y.

As mentioned earlier, we view the permissible set not as being determined by an authority but, rather, as evolving through an invisible-hand-like process with two forces: First, if the profile of intended choices from the permissible set is not feasible, then alternatives are removed or added to the permissible set. Second, when the profile of chosen alternatives is feasible, additional alternatives are added to the permissible set as long as harmony is not disturbed. Note that (y^i) can differ from (z^i), that is, when assessing the existence of a larger permissible set, choices can adapt to the loosening.

We take the permissible set to be uniform for all agents, although we are aware that there are situations in life where norms are nonuniform, such as allowing handicapped drivers to park in places where others are not permitted. The uniformity of the permissible set in our model is analogous to the uniformity of the price system in models of competitive equilibrium (although prices are often not uniform in real life). In some circumstances, uniformity can be viewed as an expression of equality of opportunity. It also is a simplicity property: in order to be followed, norms must be simple and clear, and norms are simpler when they do not distinguish between agents based on their names or preferences.

Example: A Housing Economy

Consider the housing economy with $N = \{1,2\}$, $X = \{a,b,c,d,e\}$, and preferences $a \succ^1 b \succ^1 c \succ^1 d \succ^1 e$ and $a \succ^2 c \succ^2 b \succ^2 e \succ^2 d$. One para-equilibrium is $Y = \{d,e\}$, $y^1 = d$, $y^2 = e$. This is not a Y-equilibrium since $Y = \{b,c,d,e\}$ with $y^1 = b$, $y^2 = c$ is also a para-equilibrium with a larger permissible set. The latter is the unique Y-equilibrium since the alternative a cannot be a member of any para-equilibrium permissible set as it is the top-ranked for both agents. Incidentally, the Y-equilibrium outcome is not Pareto-optimal because a is left unassigned.

Existence: Not every economy has a Y-equilibrium. In any housing economy, if at least two agents have the same strict preferences over the houses, then no Y-equilibrium exists. This is because, whatever the permissible set is, those two agents will pick the same house, which violates feasibility. This demonstrates that social norms regarding "the permissible and the forbidden" do not resolve conflicts when agents have similar preferences yet feasibility requires them to make different choices.

Example: A Single Pie

Consider the grandparents' pie economy discussed in the beginning of the chapter. There are n family members, and a pie of size 1 is to be divided among them. The set of alternatives is $X = [0,1]$ where $x \in X$ is a share of the pie. Each agent prefers to get as large a share as possible. The feasibility constraint states that the sum of their choices cannot exceed 1 (though some pie can be left over).

To see that this economy has a unique Y-equilibrium, notice first that the pair $\langle Y = [0,1/n], (y^i \equiv 1/n)\rangle$ is a para-equilibrium. There is no para-equilibrium with a point above $1/n$ in the permissible set since, then, every agent would choose a point above $1/n$, which is not feasible. Therefore, the above pair is a Y-equilibrium. There is no other Y-equilibrium since the permissible set in any para-equilibrium is a subset of $[0,1/n]$.

Example: The Quorum Economy

Consider an economy with a finite set of clubs, X. Agents have preferences over the clubs (without regard to the clubs' memberships). In order to operate, each club x needs a minimal quorum of $m_x \leq n$ (rather than having a maximal capacity as in the clubs economy). That is, feasibility requires that each club x is either empty or chosen by at least m_x members. A special case is the consensus economy where $m_x = n$ for all x, that is, feasibility requires that all agents make the same choice.

In general, if every agent were to choose his favourite club, then there would be non-empty clubs with less than a quorum. The role of the permissible set is to help the agents to coordinate their choices while imposing minimal restrictions on the permissible clubs.

A Y-equilibrium always exists: First, a para-equilibrium exists because any configuration $Y = \{x\}$ combined with all agents choosing x is a para-equilibrium. Second, since the set of subsets of X is finite, there is a para-equilibrium with a permissible set that cannot be expanded.

However, Pareto optimality is not guaranteed, as illustrated by the following example. Let $n = 6$, $X = \{a, b, c\}$, and $m_x = 3$ for all x. Two agents have the preferences $a \succ b \succ c$, two have the preferences $b \succ c \succ a$, and two have the preferences $c \succ a \succ b$. Obviously, there is no para-equilibrium with $Y = X$. Furthermore, there is no para-equilibrium with exactly two permissible clubs since four of the agents would choose one club and only two would choose the other, violating feasibility. As above, having a single club open is a para-equilibrium and since there are no multi-club para-equilibria, it is a Y-equilibrium. Thus, there are three Y-equilibria, each with a single different club open. Each Y-equilibrium outcome is not Pareto-optimal since there is an unopened club that is strictly preferred by four agents and, therefore, there is a Pareto improvement where exactly three of those four agents switch to that more-preferred club.

The Y-equilibrium concept is not meant to be normative in any sense. However, it has two fairness properties:

(i) All agents face the same choice set. Analogously, in the standard competitive equilibrium, all agents face the same trading opportunities.
(ii) It is envy-free (see Foley (1966) and Varian (1974)). Envy-freeness ensures that no agent can complain that someone else is assigned an alternative that he prefers.

Definition: Envy-freeness

A profile $(y^i)_{i\in N}$ is **envy-free** if, for all $i \neq j$, $y^i \succsim^i y^j$.

The concepts of para-equilibrium and envy-freeness are closely related. A profile is envy-free if and only if it is the outcome of some para-equilibrium: First, any para-equilibrium outcome is envy-free (no agent can envy another's choice since all agents choose from the same set). Second, if a profile (y^i) is envy-free, then $\langle\{y^1,\ldots,y^n\},(y^i)\rangle$ is a para-equilibrium.

2.2 Y-Equilibrium, Pareto Optimality, and Envy-Freeness

We have seen that Y-equilibrium profiles need not be overall Pareto-optimal. Nonetheless, they still satisfy some efficiency criterion. We now show that the Y-equilibrium profiles are precisely those which are Pareto optimal *from among the set of feasible envy-free profiles*.

Proposition 2.1: Y-equilibrium Outcome Characterization

A profile is a Y-equilibrium outcome if and only if it is Pareto-optimal among all feasible envy-free profiles.

Proof:

Let $\langle Y,(y^i)\rangle$ be a Y-equilibrium. The profile (y^i) is feasible and envy-free. If it is not Pareto-optimal among the feasible envy-free profiles, then there is a feasible envy-free profile (z^i) that Pareto-dominates (y^i). The configuration $\langle Y \cup \{z^1,\ldots,z^n\},(z^i)\rangle$ is a para-equilibrium (since $z^i \succsim^i z^j$ for all i,j, and $z^i \succsim^i y^i \succsim^i y$ for all i and $y \in Y$). By Pareto dominance, $z^i \succ^i y^i$ for at least one agent i and therefore $z^i \notin Y$. Thus, $Y \cup \{z^1,\ldots,z^n\}$ is a strict superset of Y, contradicting the definition of Y-equilibrium.

In the other direction, let (y^i) be Pareto-optimal among the feasible envy-free profiles. Let Y be the set of all elements in this profile plus any element which is weakly inferior to y^i for every agent i, namely, $Y = \bigcup_i \{y^i\} \cup \{x \mid \text{for all } i,\ y^i \succsim^i x\}$. The configuration $\langle Y,(y^i)\rangle$ is a para-equilibrium. In order to show that it is also a Y-equilibrium, we need to invalidate the existence of a para-equilibrium $\langle Z,(z^i)\rangle$ for which $Z \supsetneq Y$. If it exists, then, $z^i \succsim^i y^i$ for all i and (z^i) is envy-free. Let $x \in Z - Y$. By the definition of Y, there is an agent j for whom $x \succ^j y^j$ and, consequently, $z^j \succsim^j x \succ^j y^j$. Therefore, (z^i) is a feasible envy-free profile that Pareto-dominates (y^i), contradicting (y^i) being Pareto-optimal among the feasible envy-free profiles. Thus, no such para-equilibrium $\langle Z,(z^i)\rangle$ exists and, therefore, $\langle Y,(y^i)\rangle$ is a Y-equilibrium.

We do not take overall Pareto optimality as a necessary condition for the plausibility or desirability of a solution concept. Still, a natural question is: What condition guarantees that any Y-equilibrium outcome is overall Pareto-optimal (and not just among the envy-free profiles)? One such condition is the *imitation property*: F satisfies the imitation property if, whenever a profile is in F, so is any profile for which one agent adopts the alternative chosen by another agent instead of his own. That is, for any $(a^i) \in F$ and any $i,j \in N$, the profile where a^i is replaced with a^j is also in F. An example where the imitation property holds is the stay close economy (described in Chapter 0) since, if one agent adopts another's position, the maximal distance between any two agents does not increase.

Proposition 2.2: The Imitation Property and Pareto Optimality

Assume that F satisfies the imitation property. Then, a profile is a Y-equilibrium outcome if and only if it is overall Pareto optimal.

Proof:

Let $\langle Y,(y^i)\rangle$ be a Y-equilibrium. Assume by contradiction that there is a feasible profile (z^i) which Pareto-dominates (y^i). We construct a profile (x^i) as follows: Assign x^1, a $\succsim^1$-maximal alternative from $\{z^1,...,z^N\}$, to agent 1. Assign x^2, a $\succsim^2$-maximal alternative from $\{x^1,z^2,\ldots,z^N\}$, to agent 2, and so on. In this construction, the profile selected at each stage is feasible (due to the imitation property) and $x^i \succsim^i z^i$ for all i. Furthermore, for every agent i, the alternative x^i is $\succsim^i$-maximal from $\{x^1,\ldots,x^{i-1},z^i,\ldots,z^N\} \supseteq \{x^1,\ldots,x^N\}$. Thus, (x^i) is feasible and envy-free. It weakly Pareto-dominates (z^i) and thus Pareto-dominates (y^i), contradicting Proposition 2.1.

The other direction follows immediately from Proposition 2.1 because, under the imitation condition on F, every Pareto-optimal profile is envy-free and, therefore, is also Pareto-optimal among the envy-free allocations.

2.3 Euclidean Economies

In many common economic models, such as Walrasian economies, the set of alternatives is taken to be a subset of a Euclidean space with standard closedness, convexity, and differentiability restrictions on the alternatives, the preference relations, and the feasibility set. We now consider our framework in a Euclidean setting.

Definition: Euclidean Economy

A **Euclidean economy** is an economy $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ such that:

(i) The set X is a closed subset of some Euclidean space.

(ii) For each i, the preferences $\succsim^i$ are continuous.

(iii) The feasibility set F is anonymous (closed under permutations), compact, and contains at least one constant profile.

A Euclidean economy is **convex** if X, F, and the preferences are convex.

A Euclidean economy is **differentiable** if the preferences are strictly convex and differentiable (i.e. have differentiable utility representations or, more generally, satisfy the condition suggested in Rubinstein (2005)).

Note that in any Y-equilibrium of a Euclidean economy, the permissible set must be closed since, if $\langle Y, (y^i)\rangle$ is a para-equilibrium, then by continuity, $\langle cl(Y), (y^i)\rangle$ is also a para-equilibrium. We now show that any Euclidean economy has a Y-equilibrium.

Proposition 2.3: Existence of a Y-equilibrium in Euclidean Economies

Every Euclidean economy has a Y-equilibrium.

Proof:

Let *EFF* be the set of envy-free feasible profiles. It is non-empty because there is a feasible constant profile (which is trivially envy-free). It is closed since F is closed, and envy-freeness is defined by weak inequalities and preferences are continuous. It is compact because it is a closed subset of F, which is a compact set.

Since each $\succsim^i$ is continuous and X is a subset of a Euclidean space, there is a continuous utility function u^i representing $\succsim^i$. Thus, there is at least one profile $(z^i) \in EFF$ that maximizes $\Sigma_i u^i(x^i)$ over *EFF* and, therefore, it is Pareto optimal in *EFF*. By Proposition 2.1, (z^i) is a Y-equilibrium outcome.

2.4 The "Kosher" Economy

Previously, we discussed an economy in which one pie is allocated among a group of family members. We now consider an economy with two pies where each agent can consume a portion from *only one of the two pies*. We call it the "Kosher" economy because it reminds us of the Jewish kosher rule stipulating that diners can consume either a meat dish or a dairy dish but not both. Of course, the situation where consumption is mutually exclusive is much broader and includes, for example, a situation where the two pies stand for consumption goods in two different locations at the same time (see Malinvaud (1972)). Formally, there are two pies in the economy and at least two agents. Each agent chooses a share of a single pie, that is, X consists of all objects of the type $(a,0)$, which represents consuming a share $a \in [0,1]$ from the first pie and of the type $(0,b)$, which represents consuming a share $b \in [0,1]$ from the second pie. A bundle, which consists of a strictly positive share of each pie, is not an alternative. A profile is feasible if the sum of the agents' shares of each pie does not exceed 1. Agents have continuous and strictly monotonic preferences over X.

The Kosher economy is Euclidean and thus, by Proposition 2.3, has a Y-equilibrium. We will show that its permissible set is unique and specifies a maximal quota for each pie. Furthermore, every Y-equilibrium outcome is either Pareto optimal or "almost Pareto optimal": either both pies are fully consumed and the equilibrium outcome is Pareto optimal, or one pie is fully consumed and a small amount of the other pie is wasted.

Claim: Y-equilibrium in the Kosher Economy

In any Kosher economy:

(i) There is a unique Y-equilibrium permissible set.

(ii) In any Y-equilibrium, at least one of the pies is fully consumed.

(iii) In any Y-equilibrium, if one pie is not fully consumed, then the unallocated portion is not larger than the quota for that pie.

Proof:

By Proposition 2.3, a Y-equilibrium exists. As mentioned earlier, its permissible set must be closed. It also must include all dominated consumption bundles (otherwise such alternatives could be added in without altering the agents' choices because all preferences are monotonic). Therefore, any Y-equilibrium permissible set is specified by two quotas, a_Y and b_Y, and will be denoted by $a_Y \boxplus b_Y = \{(a,0) \mid a \leq a_Y\} \cup \{(0,b) \mid b \leq b_Y\}$.

(i) Assume not. Let $\langle a_Y \boxplus b_Y, (y^i)\rangle$ and $\langle a_Z \boxplus b_Z, (z^i)\rangle$ be two Y-equilibria with different permissible sets. The permissible sets cannot be nested. Therefore, without loss of generality, it holds that $a_Y > a_Z$ and $b_Z > b_Y$. Take the permissible set $a_Y \boxplus b_Z$ and attach to each agent an optimal bundle from $\{(a_Y,0),(0,b_Z)\}$. The number of agents assigned $(a_Y,0)$ is weakly less than the number of agents assigned $(a_Y,0)$ from $a_Y \boxplus b_Y$ because any agent i for whom $(a_Y,0) \succsim^i (0,b_Z)$, also satisfies $(a_Y,0) \succ^i (0,b_Y)$. Thus, the first pie is not over-demanded. Likewise, the second pie is also not over-demanded, and thus, there is a para-equilibrium with the permissible set $a_Y \boxplus b_Z$. This set is a strict superset of the initial two permissible sets, contradicting their being Y-equilibria.

(ii) Assume not, that is, there is a Y-equilibrium $\langle a \boxplus b, (y^i)\rangle$ where k_a agents choose $(a,0)$ while the other k_b agents choose $(0,b)$ and no pie is fully consumed, that is, $k_a a < 1$ and $k_b b < 1$.

It cannot be that $k_a = 0$ (or $k_b = 0$), since then $\langle a \boxplus 1/n, y^i = (0,1/n)\rangle$ is a para-equilibrium with a larger permissible set. Thus, both k_a and k_b are positive. Let $a' = 1/k_a > a$ and $b' = 1/k_b > b$. Consider the agents' preferences over the set $a' \boxplus b'$. If at least k_a agents weakly prefer $(a',0)$ to $(0,b')$ and at least k_b agents weakly prefer the reverse, then set $Y' = a' \boxplus b'$. If not, then without loss of generality, suppose that strictly fewer than k_a agents weakly prefer $(a',0)$ to $(0,b')$.

Let $b_\lambda = (1-\lambda)b + \lambda b'$. At least k_a agents strictly prefer $(a',0)$ to $(0,b)$. Take the maximal λ for which there are at least k_a agents who weakly prefer $(a',0)$ to $(0,b_\lambda)$, which exists because preferences are monotonic and continuous. At any $\gamma > \lambda$, fewer than k_a agents prefer $(a',0)$, so more than k_b agents strictly prefer $(0,b_\gamma)$ to $(a',0)$, and therefore by continuity more than k_b agents weakly prefer $(0,b_\lambda)$ to $(a',0)$. Set $Y' = a' \boxplus b_\lambda$.

In both cases, $Y' \supsetneq a \boxplus b$. In order to reach a contradiction we construct a para-equilibrium with permissible set being Y'. Let $N_a = \{i \mid (a',0) \succ^i (0,b_\lambda)\}$, $N_b = \{i \mid (0,b_\lambda) \succ^i (a',0)\}$ and $N_I = \{i \mid (a',0) \sim^i (0,b_\lambda)\}$. Since $|N_a| + |N_I| \geq k_a$, it follows that $|N_b| \leq k_b$. Likewise $|N_a| \leq k_a$. Thus, there is a para-equilibrium with permissible set Y' where all agents from N_a are assigned to pie A, all agents from N_b are assigned to pie B, and the agents in N_I are partitioned so that exactly k_a agents are assigned to A.

(iii) Suppose $\langle a \boxplus b, (y^i)\rangle$ is a Y-equilibrium where k_a agents choose $(a,0)$ and the leftover portion $1 - k_a a > a$. If every agent who chooses $(0,b)$ strictly prefers $(0,b)$ to $(a,0)$, then a can be slightly increased without changing consumption patterns, thus violating the maximality of the Y-equilibrium. Otherwise, there is an agent i for whom $y^i = (0,b) \sim^i (a,0)$. Then, modifying the equilibrium by assigning agent i to $(a,0)$ instead of $(0,b)$ is also a Y-equilibrium (it is a para-equilibrium and since we started with a Y-equilibrium there is no larger para-equilibrium) in which neither pie is fully consumed, contradicting (ii).

2.5 Convex Y-Equilibrium

Up until now, we have not imposed any restrictions on the structure of the permissible set. In the rest of the chapter, we will study convex Euclidean economies (which have convexity and continuity-type requirements on the set of alternatives, the preferences, and the feasible set), and we will require that the permissible set be convex.

There are two motivations for requiring a permissible set to be convex:

(i) Suppose that on a certain highway, you are told that it is permitted to drive at 20 mph and at 80 mph. Naturally, you conclude that it is also permitted to drive at 50 mph. In contrast, if you are told that it is forbidden to drive on that highway both at 20 mph and at 80 mph, you wouldn't instinctively conclude that 50 mph is also forbidden. This highlights an asymmetry between the permissible and the forbidden. Forbidden actions are usually "extreme", while permissible actions are generally a sort of "middle ground". (As always, exceptions exist: on an ice road in Estonia, it is only permitted to drive at speeds in the intervals 10–25 kph and 40–70 kph.)

(ii) As mentioned earlier, for a norm to be accepted and internalized, simplicity is a virtue. In this vein, the restriction of attention to convex permissible sets can also be viewed as a simplicity requirement. In the one-dimensional case described above, a convex permissible set is simply a minimum and maximum speed. We will demonstrate later that, in higher-dimensional spaces, the equilibrium convex permissible sets are simple in the sense that they can be described by a small number of linear inequalities.

The requirement that the permissible set is convex is similar in spirit to the standard assumption that agents choose from budget sets that are determined by common linear prices. The linearity of prices is a form of simplicity and is an attractive assumption even if in reality prices are often not linear.

Definition: Convex Y-equilibrium

A para-equilibrium $\langle Y,(y^i)\rangle$ of a convex Euclidean economy is **convex** if Y is convex. A **convex Y-equilibrium** is a convex para-equilibrium $\langle Y,(y^i)\rangle$ such that there is no other convex para-equilibrium $\langle Z,(z^i)\rangle$ with a larger permissible set $Z \supsetneq Y$.

As in the Y-equilibrium case for Euclidean economies, any convex Y-equilibrium has a closed permissible set. If not, then the closure of its permissible set, which is also convex, together with the same profile of alternatives, would constitute a convex para-equilibrium with a larger permissible set. A Y-equilibrium with a convex permissible set is a convex Y-equilibrium. However, a convex Y-equilibrium need not be a Y-equilibrium (it might be that there is no larger convex para-equilibrium permissible set, but there is a larger non-convex para-equilibrium permissible set).

2.6 Pareto Optimality and Existence of Convex Y-Equilibrium

Proposition 2.1 states that the Y-equilibrium profiles are exactly those which are Pareto-optimal among the para-equilibrium profiles (which are the feasible envy-free profiles). For convex Y-equilibria, there is a partial analogue: profiles which are Pareto-optimal among the convex para-equilibrium profiles are convex Y-equilibria profiles. However, when discussing the exchange economy, we will see that there can be convex Y-equilibrium profiles that are not Pareto-optimal among the convex para-equilibrium profiles.

Proposition 2.4: A Sufficient Condition for a Profile to be a Convex Y-equilibrium Outcome

For convex Euclidean economies, any profile which is Pareto-optimal among the convex para-equilibrium outcomes is a convex Y-equilibrium outcome.

Proof:

Given a convex Euclidean economy, let (y^i) be a convex para-equilibrium outcome that is Pareto-optimal among the convex para-equilibrium outcomes. Let P be the collection of all convex sets Y for which $\langle Y, (y^i)\rangle$ is a convex para-equilibrium. Endow P with the partial order $\supseteq$. We will use Zorn's Lemma to show that P has a maximal element. (A reminder of Zorn's Lemma: Given a partially ordered set P, if every chain — a completely ordered subset of P — has an upper bound in P, then the set P has at least one maximal element.)

Given a chain $\mathscr{C}$ of elements in P, let U be the union of the sets in $\mathscr{C}$. Clearly, U is an upper bound on $\mathscr{C}$, and we now show that U is in P. The set U is convex since for any two points $x, y \in U$, there is some $Y \in \mathscr{C}$ such that $x, y \in Y$ and, since any convex combination of x and y is in Y, it is also in U. To show that the tuple $\langle U, (y^i) \rangle$ is a para-equilibrium, it suffices to show that, for each i, the element y^i is $\succsim^i$-maximal in U. If there is an $x \in U$ such that $x \succ^i y^i$ for some i, then there is $Y \in \mathscr{C}$ such that $x \in Y$, contradicting that $\langle Y, (y^i) \rangle$ is a para-equilibrium.

Let Y^* be a maximal element of P. It is left to show that $\langle Y^*, (y^i) \rangle$ is a Y-equilibrium. Suppose that there is a convex para-equilibrium $\langle Z, (z^i) \rangle$ such that $Z \supsetneq Y^*$. It must be that $z^i \succsim^i y^i$ for all i. Since (y^i) is Pareto-optimal from among the convex para-equilibrium outcomes, it must be that $z^i \sim^i y^i$ for all i. Then, $\langle Z, (y^i) \rangle$ is also a convex para-equilibrium, contradicting the maximality of Y^*.

For Euclidean economies, we have already shown that a Y-equilibrium always exists (Proposition 2.3). The following proposition demonstrates that a convex Y-equilibrium also exists.

Proposition 2.5: Existence of a Convex Y-equilibrium

Every convex Euclidean economy has a convex Y-equilibrium.

Proof:

Let O be the set of convex para-equilibrium outcomes. The set O is not empty since F contains a constant profile $(y^i \equiv y^*)$ and the pair $\langle \{y^*\}, (y^i \equiv y^*) \rangle$ is trivially a convex para-equilibrium.

The set O is compact. To see this, since $O \subseteq F$ and F is compact, it suffices to show that O is closed. Take a sequence $\langle Y_t,(y_t^i)\rangle$ of para-equilibria such that (y_t^i) converges to (z^i) as $t \to \infty$. Let $Z \subseteq X$ be the convex hull of the limit allocations $\{z^1,\dots,z^n\}$. The configuration $\langle Z,(z^i)\rangle$ is a convex para-equilibrium since if there is an agent j and a convex combination of the $\{z^1,\dots,z^n\}$ such that $\Sigma_{i\in N}\lambda^i z^i \succ^j z^j$, then by continuity, for some large enough t, $\Sigma_{i\in N}\lambda^i y_t^i \succ^j y_t^j$. Since Y_t is convex, it holds that $\Sigma_{i\in N}\lambda^i y_t^i \in Y_t$, but this violates $\langle Y_t,(y_t^i)\rangle$ being a convex para-equilibrium.

Since O is compact, the same argument as in Proposition 2.3 implies the existence of a profile that is Pareto-optimal in O and, by Proposition 2.4, it is a convex Y-equilibrium outcome.

2.7 A Structure Theorem for Convex Y-equilibrium

Much of Economic Theory deals with establishing conditions that guarantee the existence of a solution concept. Theorems about the structure of equilibrium are less common, although, in our opinion, are more interesting. We now show that our assumptions on the economy, together with a differentiability condition, guarantee that the permissible set of convex equilibria is an intersection of at most n half-spaces (recall that n is the number of agents). Thus, the requirement that the permissible set is convex implies that the convex Y-equilibrium permissible set takes a relatively simple form.

Proposition 2.6: The Structure of Convex Y-equilibria

Let $\langle Y,(y^i)\rangle$ be a convex Y-equilibrium in a differentiable Euclidean economy. Let $J = \{i \mid y^i \text{ is not the } \succsim^i\text{-global maximum in } X\}$. Then, there is a profile of closed half-spaces $(H^j)_{j\in J}$, such that $Y = \cap_{j\in J} H^j$.

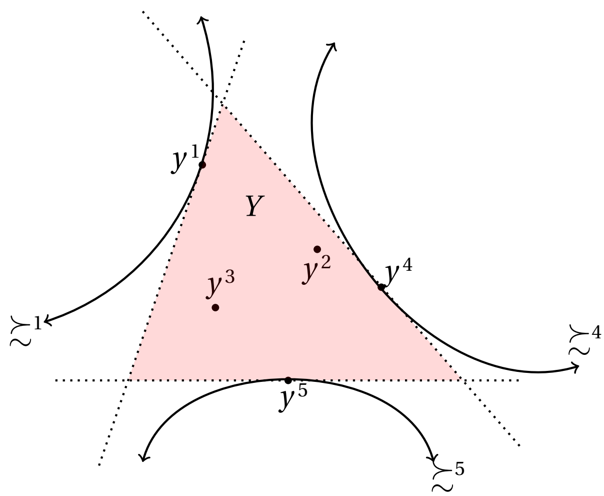


Figure 2.1 An illustration of Proposition 2.6 (note that $J = \{1, 4, 5\}$)

Proof:

First, note that if $J = \emptyset$, that is, every agent is assigned his first-best, then $Y = X$ (which is the degenerate case where Y is the intersection of an empty set of half-spaces). Otherwise, for every $j \in J$, let H^j be the unique half-space of alternatives containing y^j such that y^j is strictly preferred to all other elements in H^j. Its existence is guaranteed by the assumptions of differentiability and strict convexity of the agents' preference relations.

We first show that Y is a subset of $\cap_{j \in J} H^j$: Suppose that for some $j \in J$ there is an alternative $w^j \in Y - H^j$. By the differentiability and strict convexity of j's preferences, and for small $\varepsilon > 0$, it holds that $\varepsilon w^j + (1-\varepsilon)y^j \succ^j y^j$. By convexity of Y it holds that $\varepsilon w^j + (1-\varepsilon)y^j \in Y$. Therefore, y^j is not $\succsim^j$-maximal in Y, a contradiction.

To show that the permissible set Y is equal to $\cap_{j \in J} H^j$, it remains to be shown that $\langle \cap_{j \in J} H^j, (y^i) \rangle$ is a convex para-equilibrium. This follows from:

(i) The set $\cap_{j\in J} H^j$ is convex.

(ii) For each agent i, $y^i \in Y \subseteq \cap_{j\in J} H^j$.

(iii) For each $j \in J$, y^j is the $\succsim^j$-maximum in H^j and, thus, also in $\cap_{j\in J} H^j$.

(iv) For each $i \notin J$, y^i is the $\succsim^i$-global maximum and, thus, also in $\cap_{j\in J} H^j$.

2.8 The Division Economy

A leading economic problem is the division of a bundle among the members of a society. The grandparents single pie economy is its simplest version. The only convex Y-equilibrium is the intuitively appealing norm that forbids taking more than $1/n^{\text{th}}$ of the pie. For the multi-good division economy, the analogous norm which allows an agent to take up to $1/n^{\text{th}}$ of the total bundle is typically not a Y-equilibrium permissible set because it does not allow any trades. We proceed by exploring the properties of convex Y-equilibria in a differentiable division economy, formally defined as:

Definition: Differentiable Division Economy

A **differentiable division economy** $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ is a differentiable Euclidean economy such that:

(i) The set of alternatives is all bundles with m commodities, i.e. $X = \mathbb{R}^m_+$.

(ii) Every preference relation $\succsim^i$ is strictly monotonic (besides being continuous, strictly convex, and differentiable).

(iii) There is a bundle $e \in \mathbb{R}^m_{++}$ such that $(x^i) \in F$ if and only if $\Sigma_i x^i \leq e$.

The following claim draws a connection between convex Y-equilibrium and *egalitarian competitive equilibrium* (see Foley (1966) and Varian (1974)) which is a competitive equilibrium of the exchange economy in which each agent is initially endowed with $1/n$ of the total bundle. We will see that every egalitarian competitive equilibrium outcome is a convex Y-equilibrium outcome and, if at least one agent selects an interior bundle, then its permissible set is identical to the egalitarian competitive equilibrium's common budget set.

Claim: Egalitarian Competitive Equilibria and Convex Y-equilibria

Let $\langle p,(y^i)\rangle$ be an egalitarian competitive equilibrium in a differentiable division economy. Then, there is a convex Y-equilibrium with the same allocation $\langle Y,(y^i)\rangle$. Furthermore, if at least one of the bundles y^j is strictly positive, then Y must be $B=\{y \mid p\cdot y \le p\cdot e/n\}$.

Proof:

The pair $\langle B,(y^i)\rangle$ is a convex para-equilibrium and (y^i) is overall Pareto-optimal by the standard first welfare theorem. Thus, by Proposition 2.4, (y^i) is a convex Y-equilibrium outcome.

If $\langle Y,(y^i)\rangle$ is a convex Y-equilibrium, then by Proposition 2.6, $Y=\cap_{i\in N}H^i$, where H^i is the lower half-space of $\succsim^i$ at y^i (since no agent has his first-best, it holds that $J=N$). For all i, $B\subseteq H^i$, since otherwise there exists $z^i\in B\backslash H^i$ and, by differentiability and strict convexity, y^i would not be $\succsim^i$-optimal in B. If for some j the bundle y^j has a zero coordinate, then it can be that $B\subsetneq H^j$, but if for any j the bundle y^j is strictly positive, then $H^j=B$ and, therefore, $Y=\cap_i H^i=B$.

Comments:

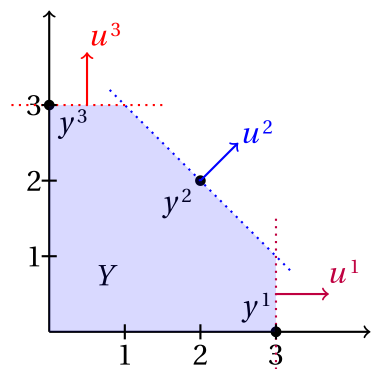
Every overall Pareto-optimal interior convex Y-equilibrium profile is an egalitarian competitive equilibrium allocation:
Let $\langle Y,(y^i)\rangle$ be a convex Y-equilibrium such that each bundle y^i is interior. By monotonicity, the alternative y^i is never $\succsim^i$-globally maximal and thus, by Proposition 2.6, $Y=\cap_{i\in N}H^i$ where H^i is the lower half-space of $\succsim^i$ at y^i and, by monotonicity, there is a positive vector p^i and a positive number w^i such that $H^i=\{x \mid p^i\cdot x\le w^i\}$. Since every y^i is interior and the allocation is Pareto optimal, the half-spaces must be parallel (otherwise, any two agents on non-parallel half-spaces could make a Pareto-improving local exchange) that is, there is a positive vector p such that $p^i=p$ for all i. It follows that $Y=\{x \mid p\cdot x\le w\}$ for some positive vector p and a positive number w. By

monotonicity, $p \cdot y^i = w$ for all i. Since $p \cdot e = p \cdot \Sigma_{i \in N} y^i = nw$, we have $p \cdot y^i = w = p \cdot (e/n)$. Thus, (y^i) is a competitive egalitarian equilibrium allocation with price vector p.

There can exist a non-interior Pareto-optimal convex equilibrium outcome that is not an egalitarian competitive equilibrium allocation:
Here is a simple example: Let $n = 3$, $m = 2$, $e = (5,5)$ and the agents' preferences be represented by the utility functions specified in Figure 2.2, panel (a) (a slight modification of the preferences will make the preference relations strictly convex):

$$u^1(x_1, x_2) = x_1$$
$$u^2(x_1, x_2) = x_1 + x_2$$
$$u^3(x_1, x_2) = x_2$$

(a) Utility functions



(b) Illustration

Figure 2.2 A convex Y-equilibrium with a non-egalitarian Pareto-optimal outcome.

Let $y^1 = (3,0)$, $y^2 = (2,2)$ and $y^3 = (0,3)$ (Figure 2.2., panel (b)). The allocation (y^i) is Pareto-optimal: If (z^i) Pareto-dominates (y^i), then $z_1^i + z_2^i \geq y_1^i + y_2^i$ for all i with at least one inequality. Thus, $\Sigma_i(z_1^i + z_2^i) > \Sigma(y_1^i + y_2^i) = 10$, which is not feasible. The set Y is the intersection of (H^i) where each H^i is a half-space of bundles below i's indifference curve, which includes y^i.

The pair $\langle Y, (y^i) \rangle$ is a convex para-equilibrium and, by Proposition 2.4, (y^i) is a convex Y-equilibrium outcome. To see this directly, note that if there were a larger convex para-equilibrium, $\langle Z, (z^i) \rangle$, then Z would contain an element that is not in Y. Any such element is strictly preferred to y^i for at least one agent i. Thus, (z^i) would Pareto-dominate (y^i).

There can exist a non Pareto-optimal interior convex equilibrium outcome: Consider the economy (depicted in Figure 2.3) with two agents, two goods, total bundle $e = (3,3)$, and kinked utility functions as depicted (a small deviation could make them strictly convex). Agent 1's indifference curve has slope -1.25, and agent 2's indifference curve has slope -0.8. The depicted allocation $y^1 = (2,1)$ and $y^2 = (1,2)$ is not Pareto-optimal since it is mutually beneficial to have agent 1 get one additional unit of good 1 and one unit fewer of good 2.

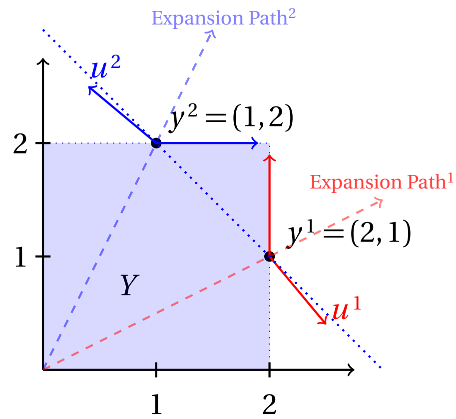


Figure 2.3 A non Pareto-optimal convex equilibrium.

The configuration $\langle Y,(y^i)\rangle$ is a convex para-equilibrium. In any larger convex para-equilibrium, $\langle Z,(z^i)\rangle$, the convex set Z includes a bundle that is strictly better for at least one of the agents and, therefore, $z^1 \neq y^1$ and $z^2 \neq y^2$. Given the agents' indifference curves, it must be that in z_1 agent 1 receives more than 2 units of good 1 . Then, it must be that $z^1_1 + z^1_2 > 3$ since if $z^1_1 + z^1_2 \leq 3$ then the intersection point of the line between z and $(2,2)$ (which has a slope of at most -2 which is smaller than -1.25) and the expansion ray of agent 1 is strictly preferred by agent 1 to z. Likewise $z^2_1 + z^2_2 > 3$, a contradiction.

Initial Endowments: Recall that a division economy differs from the standard exchange economy as it does not specify an initial distribution of the goods. One way to incorporate initial endowments into our framework is by the following notion of a *trade economy*. Let (e^i) be an initial endowment profile. Let $X = \mathbb{R}^m$ where a member of X is interpreted as a trade (and thus includes negative components as well). Set F to include all profiles of trades (t^i) such that $\Sigma_i t^i = 0$ and for every agent i, the post-trade bundle $t^i + e^i \geq 0$. As to the preferences, assume that each agent i has a basic preference relation $\succsim^i_c$ over the set of bundles (satisfying the standard division economy assumptions).

Among trades that give an agent a non-negative amount of every good, agent i's preferences $\succsim^i$ on X are induced from their basic preferences by $t^i \succsim^i s^i$ if $t^i + e^i \succsim^i_c s^i + e^i$. Every agent prefers the no-exchange option 0 to any trade which leaves them with a negative amount of any good.

Analogous results to the previous claims for the division economy also hold for the trade economy: (i) the profile of trades in any competitive equilibrium in the standard exchange economy is a convex Y-equilibrium outcome in this trade economy, and (ii) any Pareto-optimal convex Y-equilibrium outcome in the trade economy, where at least one agent has a strictly positive post-trade allocation, is a profile of trades in a competitive equilibrium of the standard exchange economy.

2.9 The Give-and-Take Economy

Recall that in the give-and-take economy, the set of alternatives is $X = [-1, 1]$, where a positive x represents a withdrawal of x from a social fund while a negative x represents a contribution of $-x$. Feasibility requires that the social fund be balanced, that is, $(x^i) \in F$ iff $\Sigma_i x^i = 0$. All agents have strictly convex preferences over X with agent i's ideal denoted by $peak^i$. As mentioned earlier, the give-and-take economy is an economic situation in which the market plays no role. We will see that norms regarding what is permissible and what is forbidden can serve as an effective non-market tool for achieving harmony.

The case $\Sigma_i peak^i = 0$ is "bliss": everything is permitted and $\langle X, (peak^i) \rangle$ is a convex Y-equilibrium. However, in general, there is tension between feasibility and the agents' desires. The following claim characterizes the convex Y-equilibrium for the case where the sum of what people ideally want to take is greater than what people ideally want to give. We will see now that, in this case, there is a unique convex Y-equilibrium. In it, people are allowed to give as much as they want but there is a bound on the maximum that can be taken, and its outcome is Pareto optimal.

Claim: A Characterization of the Convex Y-equilibrium

Consider a give-and-take economy with $\Sigma peak^i > 0$. There is a unique convex Y-equilibrium $\langle Y, (y^i)\rangle$. The set Y takes the form $[-1, m]$ for some $m > 0$, and (y^i) is Pareto optimal.

Proof:

Consider a permissible set of the form $[-1, m]$. If $m < 0$, then all agents must give. If $m \geq 0$, then every agent who wants to give will select his peak, and every agent who wants to take is either at his peak or has a peak to the right of m and makes do with taking m. Let $D(m)$ be the sum of all agents' choices given the permissible set $[-1, m]$. The function D is continuous, strictly increasing for any m smaller than $\max\{peak^i\}$, and is constant with value $\Sigma_i peak^i > 0$ for any larger m. In particular, $D(0) \leq 0$ and $D(1) > 0$. Thus, there is a unique $m^* \geq 0$ for which $D(m^*) = 0$.

The permissible set $[-1, m^*]$, together with the agents' optimal choices from that set, constitutes a convex para-equilibrium. It is also a convex Y-equilibrium because there is no convex para-equilibrium with a larger permissible set. If there were, it would have the form $[-1, m]$ where $m > m^*$, but then agents would take too much (since $D(m) > 0$).

The profile (y^i) is Pareto optimal: For each i, y^i is at or to the left of his peak. Thus, if $(z^i) \in F$ Pareto-dominates (y^i), then $y^i \leq z^i$ for all i with at least one strict inequality, thus $0 = \Sigma y^i < \Sigma z^i$, violating feasibility.

To prove uniqueness of the convex equilibrium, it remains to be shown that any closed convex para-equilibrium permissible set $[x, y]$ is included in $[-1, m^*]$. In order for the social fund to be balanced, it must be that $x \leq 0 \leq y$. In equilibrium, agents who wish to give will do so at either their peak or at x if $peak^i < x$. Therefore, the total giving in $[x, y]$ is not more than that in $[-1, m^*]$. Since the social fund is balanced, the total taking in $[x, y]$ must also be less than or equal to that in $[-1, m^*]$, and therefore $y \leq m^*$. Thus, $[x, y] \subseteq [-1, m^*]$.

Comment: For this economy, while convex Y-equilibria are Pareto-optimal, a Y-equilibrium outcome need not be. A detailed example appears in Richter and Rubinstein (2020). The essence of the example is as follows: Let $X=\{-2,-1,0,1,2\}$ and $n=2$. The agents' "convex" preference relations are $1 \succ^1 0 \succ^1 -1 \succ^1 -2 \succ^1 2$ and $1 \succ^2 2 \succ^2 0 \succ^2 -1 \succ^2 -2$. The "convex" permissible set $Y=\{-2,-1,0\}$, together with the profile $y^1=y^2=0$, is a "convex" Y-equilibrium with a Pareto-optimal outcome. However, it is easy to verify that the non-convex permissible set $Y=\{-2,2\}$ with the profile $y^1=-2, y^2=2$ is a Y-equilibrium whose outcome is Pareto dominated by $z^1=-1$, $z^2=1$.

2.10 The Stay Close Economy

The stay close economy is a convex Euclidean economy in which X is a closed convex set of locations and F is the set of profiles for which the distance between any two agents is at most d^*. That is, each member of the group chooses a position (for example, a political stance or a geographical location), and the group's survival requires that the members "stay close" to each other. As always, each agent has strictly convex preferences for his own location without regard to the location of others. The potential source of conflict is that the group members have a diverse set of ideal locations which fails the closeness requirement. Note that the set F satisfies the imitation condition defined in Section 2.3. When $d^*=0$, this economy is called a *consensus economy*.

In a centralized society, the authorities can coerce agents into occupying locations that guarantee survival. In a market, members would have to pay each other to stay close by. The Y-equilibrium idea is that there are norms that determine the borders of the permissible locations and strike a balance between societal harmony and individual liberty. Each agent chooses his most preferred location within the borders, and the outcome is that they all live close enough to one another. The borders are maximally liberal in the sense that if the borders are enlarged in any way, then the resulting individual choices would not be "close enough".

A modified serial dictatorship provides a simple method for finding a Pareto-optimal profile and proving its existence: agent 1 selects his ideal point $x^1 = peak^1$ in X, then agent 2 selects his most preferred point from among those that are "close enough" to x^1, and each subsequent agent i selects his most preferred point x^i from among those which are "close enough" to all of those previously selected, that is, $\{x \mid d(x, x^j) \leq d^*$, for every $j < i\}$.

The following claim establishes that every Pareto-optimal profile is both a Y-equilibrium outcome and a convex Y-equilibrium outcome. As a partial converse, it also shows that Y-equilibrium outcomes are Pareto optimal; however, convex Y-equilibrium outcomes may or may not be.

Claim: Pareto Optimality and Y-equilibrium

For a stay close economy:

(i) A Y-equilibrium and a convex Y-equilibrium exist. Moreover, any Pareto-optimal allocation is both a Y-equilibrium outcome and a convex Y-equilibrium outcome.

(ii) Every Y-equilibrium outcome is Pareto optimal. However, a convex Y-equilibrium outcome need not be.

(iii) If X is a subset of a one-dimensional Euclidean space, then any convex Y-equilibrium outcome is Pareto optimal.

Proof:

(i) The modified serial dictatorship algorithm above establishes the existence of Pareto-optimal allocations. Given a Pareto-optimal allocation (y^i), let Y be the convex hull of $\{y^1, \ldots, y^n\}$. Any point $\Sigma_j \lambda_j y^j$ in Y is at most d^* away from every y^i since $d(\Sigma_j \lambda_j y^j, y^i) \leq \Sigma_j \lambda_j d(y^j, y^i) \leq \Sigma_j \lambda_j d^* \leq d^*$ (the first inequality is due to the triangle inequality). Each y^i is $\succsim^i$-maximal in Y since any agent moving to another location in Y preserves feasibility and (y^i) is Pareto optimal. Therefore, $\langle Y, (y^i) \rangle$ is a convex para-equilibrium. By Proposition 2.1, (y^i) is a Y-equilibrium profile and, by Proposition 2.4, it is also a convex Y-equilibrium profile.

(ii) Since F satisfies the imitation property, by Proposition 2.2, any Y-equilibrium outcome is Pareto optimal. The following economy shows that a convex Y-equilibrium can be non Pareto-optimal: Let $n = 2$, $d^* = 0$, $X = \mathbb{R}^2$, and agents' preferences be given by $U^1(x_1, x_2) = 2x_2 - (x_2 - x_1)^2$ and $U^2(x_1, x_2) = 2x_2 - (x_2 + x_1)^2$ (see Figure 2.4). From $Y = \{(x_1, x_2) \mid x_2 \leq 0\}$, both agents choose $y^1 = y^2 = (0,0)$ and the pair $\langle Y, (y^i)\rangle$ is a convex para-equilibrium. If there were a larger convex para-equilibrium set, then there would be one of the form $Z = \{(x_1, x_2) \mid x_2 \leq z\}$ with $z > 0$. From Z, agent 1 prefers (z, z) and agent 2 prefers $(-z, z)$, and this profile is not in F. The equilibrium outcome is not Pareto optimal since both agents prefer $(0,1)$ to $(0,0)$. This example can be easily modified for any $d^* > 0$ by setting $Y = \{(x_1, x_2) \mid x_2 \leq d^*/2\}$, $y^1 = (d^*/2, d^*/2)$ and $y^2 = (-d^*/2, d^*/2)$.

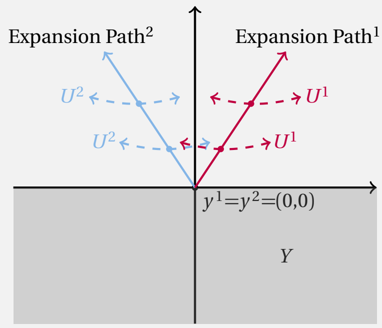


Figure 2.4 Non Pareto-optimal convex equilibrium in a consensus economy

(iii) Let $\langle Y, (y^i)\rangle$ be a convex Y-equilibrium, L be the minimum of the agents' peaks, R be the maximum, $\underline{y} = \min_i y^i$, and $\overline{y} = \max_i y^i$.

If $R - L \leq d^*$, then $Y = X$ and every agent chooses his peak, which is obviously a Pareto-optimal outcome.

If $R - L > d^*$, then it must be that $\overline{y} - \underline{y} = d^*$, since otherwise $\overline{y} - \underline{y} < d^*$, and there is an agent who is not at his peak. Thus, his choice must be on the boundary of Y. This boundary can be slightly enlarged and the profile of agents' new optimal choices will be feasible.

By the convexity of Y, each agent who chooses $\overline{y}$ is at his peak or wants to move to the right, each agent who chooses $\underline{y}$ is at his peak or wants to move to the left, and the others choose their peaks. Thus, any profile that Pareto dominates (y^i) must increase the maximum distance between agents, which violates feasibility since $\overline{y} - \underline{y} = d^*$.