NO PRICES NO GAMES!

FOUR ECONOMIC MODELS

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3 Status and Indoctrination

We now turn to a different notion of economic harmony studied in Richter and Rubinstein (2015). As always in this book, an equilibrium is a *profile* of alternatives (one for each agent) and an additional parameter. Here, the parameter is a commonly accepted ordering on the set of all alternatives that affects the choices agents can make. We refer to it as a *public ordering*.

We have in mind three interpretations of a public ordering:

Values. When the alternatives are objects, the public ordering may reflect their "value" or "worth". A holder of an object can exchange it for any lower-valued object, but not for a higher-valued one. This interpretation aligns with the standard notion of the "more expensive" relation — a holder of a bundle in a market can exchange it for a cheaper one, but not for a more expensive one.

Prestige. When the alternatives are positions in a society, the public ordering may reflect the prestige of these positions. According to this interpretation, an agent can exchange his position for any less prestigious one, but not a more prestigious one.

Under the above two interpretations, society restricts an agent's *ability* to replace the alternative he has. He can only move "down" to a "less valuable" or "less prestigious" one, but not "up". In contrast, in the next interpretation, the public ordering's meaning is reversed: lower-ranked alternatives are more "valuable" for society.

Indoctrination. Agents are indoctrinated by society regarding the interests of society as a whole. A public ordering inversely ranks the alternatives according to their benefit to society: the lower ranked an alternative is, the more beneficial it is to society. An agent is only willing to replace his assigned alternative with one that is better for society (i.e. lower-ranked by the public

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ordering). The indoctrination does not affect the agent's basic preferences (in contrast to the biased preferences model discussed in Chapter 4) but, rather, modifies his choice set. An agent only considers moving from one alternative to another if it benefits society (and will only make such a move if he also personally benefits).

With the above interpretations in mind, we will discuss two types of equilibria, which fit into the two categories of equilibrium discussed in Section 0.4. The first is the *status equilibrium*: it is a feasible profile of choices and a public ordering such that no agent strictly prefers any alternative that is weakly lower-ranked by the public ordering than the one assigned to him. Thus, the public ordering limits the agents' deviations from the equilibrium profile: an agent who is assigned an alternative only considers deviating to alternatives that are weakly lower-ranked (by the public ordering) than the one he is assigned. Deviations are purely self-serving and contemplated without regard to feasibility.

In the taxonomy of Section 0.4, the status equilibrium belongs to the deviation group (like Nash equilibrium). In the last section of the chapter, we will study the *initial status equilibrium* concept, which fits into the choice group (like competitive equilibrium). This concept operates on an extended economy (in which the model of an economy is extended to include an initial profile). In the initial profile, each agent is assigned an alternative that he can always choose and which, together with the public ordering, determine the agent's choice set. An initial status equilibrium consists of a feasible profile and a public ordering, but this time the profile of choices must be such that no agent strictly prefers any alternative that is weakly lower-ranked by the public ordering than the one initially assigned to him. Thus, the public ordering limits an agent's choice set: he only considers alternatives that are weakly lower-ranked than his initial alternative.

3.1 Status Equilibrium

Definition: Status Equilibrium

Given an economy $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$, a **status equilibrium** is a pair $\langle P, (x^i)_{i\in N}\rangle$ where $(x^i)_{i\in N}$ is a profile and P is an ordering (a complete, reflexive, and transitive binary relation) on X satisfying:

Feasibility: the profile (x^i) is in F.
Personal optimality: for every agent i, the element x^i is $\succsim^i$-maximal in $\{z \in X \mid x^i P z\}$.

The ordering P is referred to as a **public ordering**.

As mentioned, under the first two interpretations of a public ordering, it ranks the alternatives by value or prestige. The term aPb means that *a is more expensive than b* or that *a is more prestigious than b*. An equilibrium public ordering stabilizes the equilibrium profile in the sense that every agent is satisfied with his assignment *given* that he is bounded by the worth (or prestige) of his assigned alternative. Under the third interpretation, P is a social motive that systematically affects an agent's willingness to exchange his assigned alternative. The term aPb means that "*a* is *less socially desirable* than *b*" (this is not a mistake... *less* and not *more*). If an agent i is assigned x^i, then he cannot bear the idea of exchanging it for an alternative that is less socially desirable and, therefore, he only considers more socially desirable alternatives (i.e. those which are lower-ranked by P). Under this interpretation, an equilibrium consists of a public ordering and a feasible profile in which no agent both: i) wishes to exchange his assigned alternative, according to his personal preferences and ii) is able to justify the exchange as furthering society's interests, according to the public ordering.

Proposition 3.1: A Second Welfare Theorem

Any Pareto-optimal profile is a status equilibrium profile.

Proof:

Let (a^i) be a Pareto-optimal profile. Define the binary relation D on $A = \{a^1, \ldots, a^n\}$ by xDy if x is desired by a holder of y, that is, there are i and j such that $x = a^i \succ^j a^j = y$. If D has a cycle, then there is a set of agents who can permute their alternatives among themselves (recall that F is closed under permutations) so that all of them are strictly better off, contradicting (a^i) being Pareto-optimal. Since D has no cycles, it can be extended to a complete ordering over A. Then, D can be extended to a strict ordering P on the entire set X by putting all elements in $X - A$ above all elements in A (making all unassigned elements "unaffordable") and arbitrarily ranking the elements in $X - A$ among themselves. Personal optimality holds since, for every agent i, the alternative a^i is optimal in $\{x \mid a^i P x\}$ (if $a^i P x$, then $x = a^j$ for some j, and if i were to prefer it, then xDa^i, which contradicts $a^i P x$ since P extends D).

By the same proof, any feasible profile (Pareto-optimal or not) for which the relation D does not have cycles is a status equilibrium profile. In particular, in the consensus economy, where all agents have to make the same choice, any profile that assigns the same element x^* to all agents is supported by any public ordering that ranks x^* as the unique lowest element in X and thus all other alternatives are "blocked". Such a profile might be not Pareto-optimal. Thus, any Pareto-optimal profile is a status equilibrium profile, but a status equilibrium profile does not have to be Pareto-optimal.

3.2 Status Equilibrium – Examples

Example: The Jobs Economy

Let X be a non-singleton set of types of jobs. Each agent holds *strict* preferences on X. Feasibility is given by a vector $(n_x)_{x \in X}$ where n_x is the number of available jobs of type x (non-emptiness of F requires that $\Sigma_{x \in X} n_x \geq n$).

A public ordering in this example has a natural interpretation of social status, which is often associated with a job. Once an agent is assigned to a job, he cannot switch to a higher-status job but he can switch to any job of equal or lower status (e.g. a professor can move to a lower-ranked university but not to a higher-ranked one). The housing model of Shapley and Scarf (1974) is the special case where $n_x \equiv 1$ and $|X| = n$.

Claim: The following holds for the jobs economy:
(i) If $\Sigma_x n_x = n$, then the First Welfare Theorem holds: every status equilibrium profile is Pareto-optimal.
(ii) If $\Sigma_x n_x > n$, then the First Welfare Theorem fails: there is always a status equilibrium profile that is not Pareto-optimal.

Proof: (i) Let $\langle P, (x^i)\rangle$ be a status equilibrium. Assume by contradiction that the feasible profile (y^i) Pareto-dominates (x^i). Let j be an agent for whom x^j is P-maximal from among $\{x^i \mid y^i \neq x^i\}$. Since preferences are strict, it must be that $y^j \succ^j x^j$ and, therefore, $y^j P x^j$. Since $\Sigma_x n_x = n$, it must be that in any feasible profile, all jobs are filled. Therefore, there is another agent whose original job is y^j and whose new job is not, contradicting the P-maximality of x^j from among $\{x^i \mid y^i \neq x^i\}$.

(ii) Let (x^i) be a Pareto-optimal profile. By Proposition 3.1, there is a public ordering such that $\langle P, (x^i)\rangle$ is a status equilibrium. Let z denote a job with spare capacity, and let j be an agent who does not have job z (which exists since $\Sigma_x n_x > n$ and $|X| > 1$). Let (y^i) be the feasible profile obtained from (x^i) by moving j from x^j to z. Since (x^i) is Pareto-optimal, every agent who does not have job z strictly prefers his assigned job to z and thus, (y^i) is not Pareto-optimal. Let P' be the public ordering obtained from P by moving z to the bottom rank. The pair $\langle P', (y^i)\rangle$ is clearly a status equilibrium.

Example: R-Monotonic Preferences

Let R be a strict partial ordering (irreflexive, transitive, and anti-symmetric but not necessarily complete) on X. A preference relation $\succsim$ is **R-monotonic** if $a \succ b$ whenever aRb.

For example, let X be a set of bundles and R be defined by xRy if the bundle x contains weakly more than y of every good and strictly more of at least one. In this case, R-monotonicity is the standard notion of strong monotonicity.

It will now be shown that, for any economy with R-monotonic preferences, any status equilibrium profile can also be supported as a status equilibrium with an R-monotonic public ordering. Thus, a stronger assumption on agents' preferences (R-monotonicity) leads to stronger conclusions about the equilibrium public ordering (being R-monotonic).

Claim: Let R be a strict partial ordering and let $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ be an economy where every preference $\succsim^i$ is R-monotonic. If $\langle P, (x^i)_{i\in N}\rangle$ is a status equilibrium, then there is an R-monotonic ordering Q such that $\langle Q, (x^i)_{i\in N}\rangle$ is also a status equilibrium.

Proof: Define the *desire* binary relation D as yDz if there is an agent who is assigned z and strictly prefers y (and thus, it must be that yPz strictly). Let $S = R \cup D$. The relation S is acyclic: if not, let $z_1 S_1 z_2 S_2 z_3 S_3 \ldots z_m S_m z_1$ be a minimal cycle where each S_i is either R or D.

- It cannot be that all S_i are R because R is acyclic.
- It cannot be that all S_i are D since $z_{i-1} D z_i$ implies $z_{i-1} P z_i$ strictly and thus, a D-cycle implies a strict P-cycle, which is impossible.
- It cannot be that the cycle $S_1, \ldots, S_m$ contains both D and R. This is because, if it did, then it would contain an R followed by a D. However, if $aRbDc$, then there is a j such that $c = x^j$ and $b \succ^j c$. Since $\succ^j$

extends R, it follows that $a \succ^j b$ and therefore, $a \succ^j c$ and so aDc. Therefore, the cycle can be shortened.

Thus, S is a strict partial ordering. Extend S to an ordering Q. Since S extends R, so does Q and thus, Q is R-monotonic. Since S extends D, so does Q and thus, $\langle Q,(x^i)\rangle$ is a status equilibrium.

3.3 A Detour: Convex Preferences

In Section 3.4, we will refine the notion of a status equilibrium by imposing some structure to the public ordering. In preparation, we make a detour to the concept of convex preferences.

One conventional definition of convex preferences for Euclidean spaces requires that if a is weakly preferred to b, then any convex combination of a and b is also weakly preferred to b. This definition is equivalent to requiring that all upper contours (sets of the type $\{x \mid x \succ a\}$) are convex sets. Both of these definitions refer to the term “convex combination”, which itself uses an algebraic structure on the space of alternatives and so does not apply to economies where the set X lacks such a structure.

Following Richter and Rubinstein (2019), we suggest an alternative definition of convex preferences which generalizes the standard Euclidean notion and is also applicable to spaces without algebraic structure. A cornerstone of this approach is the view that preferences are built from primitive building blocks. Here, we take the building blocks to be the members of a set of orderings Λ, which we call *primitive orderings*. Each primitive ordering is a complete, reflexive, and transitive binary relation over the set X (indifferences are allowed). We interpret the primitive orderings as expressions of objective attributes of the alternatives that are in the vocabulary of all agents.

The assumption behind this definition is that, when thinking about replacing an alternative $b \in X$, an agent has in mind a necessary criterion (primitive ordering) that is *critical*, in the sense that, for an alternative to be better than b, it must be better by this criterion. Note that the critical criterion can depend on b.

For example, imagine a department chair who is contemplating replacing b, who is a weak teacher. In this case, the critical consideration may be pedagogical ability, and any teacher who is pedagogically worse than b will be rejected. However, this does not mean that any candidate who is pedagogically better than b will be preferred. Again, the critical criterion can vary from one alternative to another: when the department chair considers replacing c, who is a great teacher and a poor researcher, he may feel that research ability is now critical, and thus, any candidate who is a worse researcher than c will be judged to be a worse candidate than c.

Definition: Λ-convex Preferences

Let X be a set of objects and Λ be a set of orderings on X referred to as **primitive orderings**. The symbol $\trianglerighteq$ represents a generic member of Λ.

A preference relation $\succsim$ on X is **Λ-convex** if:

$\forall b \in X, \exists \trianglerighteq \in \Lambda$ such that for $x \neq b$ it is necessary for $x \succ b$ that $x \triangleright b$.

A preference relation $\succsim$ on X is **Λ-strictly convex** if:

$\forall\, b \in X, \exists \trianglerighteq \in \Lambda$ such that for $x \neq b$ it is necessary for $x \succsim b$ that $x \triangleright b$.

In both definitions, the ordering $\trianglerighteq$ is called a **critical direction** at b; (there can be multiple critical directions).

Three comments:

(i) Every (strict) primitive ordering in Λ is Λ-(strictly) convex: for each alternative, the primitive ordering itself is a critical direction.

(ii) A "Pareto" property holds: If b and c are distinct, $b \trianglerighteq c$ for every $\trianglerighteq \in \Lambda$, and $\succsim$ is Λ-convex, then $b \succsim c$. This is because there is a critical ordering $\trianglerighteq$ attached to b and $b \trianglerighteq c$ and therefore, c cannot be strictly preferred to b. For Λ-strictly convex preferences, the conclusion is stronger, namely, $b \succ c$.

(iii) In Richter and Rubinstein (2019), we also suggested other similar definitions of convex preferences and discussed their connection to Edelman and Jamison (1985)'s notion of "abstract convexity".

Underpinning our convexity notion is the abstraction of a concept that plays a fundamental role in economic analysis when we talk about convex preferences on a Euclidean space: for each alternative, there is a *hyperplane* which contains it, such that all weakly preferred alternatives lie on one side of the hyperplane. In the same spirit, our notion of convex preferences requires that for every alternative there is a primitive *ordering* that puts all preferred alternatives on one side of the ordering.

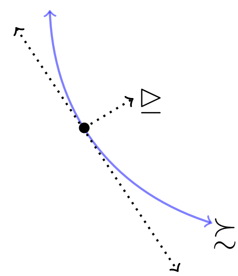

Figure 3.1 A supporting hyperplane and its corresponding critical direction

The definition of convex preferences is attractive for several reasons:

(a) It is compelling as a procedural assumption of preference formation.

(b) It emphasizes and allows for the dependence of the convexity property on the specification of the considerations used to construct preferences.

(c) It does not require any algebraic structure.

(d) For continuous preferences on Euclidean spaces, it generalizes the standard convexity notion as shown in the following example:

Example: Euclidean Space with Algebraic Linear Orderings

Let X be an open convex subset of a Euclidean space. For any vector $v \neq 0$, define the *algebraic linear ordering* $\geq_v$ by $x \geq_v y$ if $v \cdot x \geq v \cdot y$. Let Ψ be the set of all algebraic linear orderings.

Claim: Let $\succsim$ be a continuous preference relation on X. Then: $\succsim$ is convex by the standard definition if and only if $\succsim$ is Ψ-convex.

Proof: Assume $\succsim$ is convex by the standard definition, i.e. for every $b \in X$, the set $U_{\succ}(b) = \{z \mid z \succ b\}$ is convex. It is also open, since $\succsim$ is continuous. By the separating hyperplane theorem, there exists $\geq_v \in \Psi$ such that $x >_v b$ for every $x \in U_{\succ}(b)$. That is, $\geq_v$ is a critical direction.

Assume $\succsim$ is Ψ-convex. Let a, $c \in X$ such that $a, c \succ b$, and let z be an element on the line between a and c. By Ψ-convexity, there is a critical direction $\geq_v$ at z. Then, $z \geq_v a$ or $z \geq_v c$ or both, and since $\succsim$ is Ψ-convex, it follows that $z \succsim a$ or $z \succsim c$ or both and thus $z \succ b$. ■

The above example can be applied to the one-dimensional case. There, the set Ψ consists of two orderings $\trianglerighteq_L$ (which ranks left alternatives higher) and $\trianglerighteq_R$ (which ranks right alternatives higher). The claim demonstrated that for the one-dimensional case, Ψ-convexity of continuous preferences is equivalent to standard convexity of preferences, namely any such preferences have a maximal region (which may be an interval or a single element), the preferences are weakly increasing up to this region, and weakly decreasing beyond it. The following example shows that strict Ψ-convexity is equivalent to single-peakedness of preferences.

Example: Left and Right

Let $X = [0,1]$, and suppose that Λ contains two orderings: the rightist $\trianglerighteq_R$ (which ranks elements to the right higher) and the leftist $\trianglerighteq_L$ (which ranks elements to the left higher). A preference relation is *single-peaked* if:

(i) it has a unique maximum point (*peak*) in X; and

(ii) it is strictly increasing below the peak and strictly decreasing above it.

Claim: Let $\Lambda = \{\trianglerighteq_L, \trianglerighteq_R\}$ and $X = [0,1]$. A continuous preference relation is Λ-strictly convex if and only if it is single-peaked.

Proof: Suppose $\succsim$ is singled-peaked. At any $b > peak$, the ordering $\trianglerighteq_L$ is critical, while at any $b < peak$ the ordering $\trianglerighteq_R$ is critical. At the peak, both orderings are critical.

Suppose $\succsim$ is Λ-strictly convex. Since the preferences are continuous and X is compact, there is a $\succsim$-maximal element. Suppose that there are two $\succsim$-maximal elements $x < z$, and let $y \in (x,z)$. The ordering $\trianglerighteq_L$ is not a critical direction at y because $z \succsim y$ and $z \ntrianglerighteq_L y$. Likewise, $\trianglerighteq_R$ is not a critical direction at x because $x \succsim y$ and $x \ntrianglerighteq_R y$. Therefore, there is no critical direction at y, a contradiction to the Λ-strictly convexity of $\succsim$.

Let M be the $\succsim$-maximal element. For every $y < x < M$, the critical ordering at x must be $\trianglerighteq_R$ and thus $y \prec x$. Therefore, $\succsim$ is strictly increasing to the left of M. Likewise, for every $x > M$, the critical ordering must be $\trianglerighteq_L$, and $\succsim$ is strictly decreasing to the right of M. Thus, the preferences are single-peaked with the peak at M. ■

We now illustrate the richness of the Λ-convexity notion by considering its application to the case of preferences over collections of distinct goods.

Example: Collections

Let Z be a set of distinct indivisible goods and X be the set of all collections of items from Z. Here, unlike in the housing economy, an agent can have more than one good or none at all. We will use the Greek symbols Θ and Φ for collections of goods.

For every v, a non-negative function on Z, let $\trianglerighteq_v$ be the ordering of X represented by the utility function $v(\Theta) = \Sigma_{z\in\Theta} v(z)$. That is, $v(\Theta)$ is the sum of the v-values attached to the individual items in the set Θ. Let Λ be the set of such orderings. An interpretation of these orderings is that a value is attached to *each good* and the value of a collection is the sum of the values of the goods in the collection.

Claim: A preference is Λ-convex if and only if it is weakly monotonic with respect to the inclusion relation.

Proof: Let $\succsim$ be a Λ-convex preference relation. Take any two sets for which $\Phi \supset \Psi$. For any v, the larger set will be ranked higher by v. By the "Pareto" property $\Phi \succsim \Psi$ (see comment (ii) after the definition of Λ-convex preferences). Thus, $\succsim$ is weakly monotonic.

On the other hand, let $\succsim$ be a weakly monotonic preference. Given a collection Φ, define v by

$$v(z) = \begin{cases} 0 & z \in \Phi \\ 1 & z \notin \Phi \end{cases}$$

If $\Psi \succ \Phi$, then by weak monotonicity, there must be a $z \in \Psi - \Phi$, and therefore, $v(\Psi) \geq 1 > 0 = v(\Phi)$. Therefore, $\geq_v$ is a critical ordering at Φ. ■

Construction of Convex Preferences for Finite Sets: For a finite set X, if all orderings in Λ are strict, then the following procedure builds a Λ-convex preference relation: Take an alternative x_1 which is at the bottom of one of the primitive orderings, and place it at the bottom of $\succsim$. Then, let x_2 be an alternative at the bottom of $X - \{x_1\}$ with respect to one of the primitive orderings, and place it (strongly or weakly) above x_1. Continue this procedure until all alternatives are exhausted. The constructed preference relation is Λ-convex since the position of each $b \in X$ in $\succsim$ was determined when b was at the bottom of some primitive ordering, which is then a critical direction at b since any strictly preferred alternative is ranked strictly higher than b by that ordering.

If X is finite and all orderings in Λ are strict, then every Λ-convex preference relation $\succsim$ can be constructed by the procedure described above:

To apply the construction to obtain $\succsim$, at every stage we must identify an alternative and a primitive ordering $\trianglerighteq$ so that the alternative is both $\succsim$-minimal and $\trianglerighteq$-minimal from among the remaining alternatives. To start, pick $x \in X$ which is $\succsim$-minimal, and let $\trianglerighteq \in \Lambda$ be a critical direction at x. If x is $\trianglerighteq$-minimal, then set $x_1 = x$. If not, then pick y which is minimal according to the same $\trianglerighteq$. The alternative y is also $\succsim$-minimal since $\trianglerighteq$ is a critical direction at x and $x \trianglerighteq y$, and then set $x_1 = y$. Continue inductively with the remaining alternatives.

Utility Representation: We say that a preference relation $\succsim$ over X has a Λ-*maxmin representation* if there is a profile of functions $(U_{\trianglerighteq})_{\trianglerighteq \in \Lambda}$ such that for every $\trianglerighteq \in \Lambda$ the function $U_{\trianglerighteq}$ is a utility representation of $\trianglerighteq$ and the function $U(x) = \min_{\trianglerighteq} U_{\trianglerighteq}(x)$ is well-defined and represents $\succsim$.

If Λ is finite and $\succsim$ has a Λ-maxmin representation $(U_{\trianglerighteq})$, then $\succsim$ is Λ-convex: For any $b \in X$, take the ordering $\trianglerighteq \in \Lambda$ for which $U_{\trianglerighteq}(b)$ is minimal. The ordering $\trianglerighteq$ is a critical direction at b because $b \trianglerighteq x$ implies $U_{\trianglerighteq}(b) \geq U_{\trianglerighteq}(x)$ and thus, $U(b) \geq U(x)$ and $b \succsim x$. In Richter and Rubinstein (2019), it is shown that for finite X, the converse is also true: any Λ-strictly convex preference relation has a Λ-maxmin representation.

The existence of such a representation means that we can identify every alternative in the set X by a vector of numbers in $\mathbb{R}^{\Lambda}$ such that:
(i) for every primitive ordering, the values that are attached to the elements in X at the corresponding coordinate are consistent with that primitive ordering's ranking; and
(ii) the preferences are represented by the minimum value attached to an alternative across the different dimensions.

3.4 Primitive Equilibrium

In the canonical consumer model, the set of alternatives (bundles) is a subset of a Euclidean space and the following holds:

(i) Agents have standard convex preferences.
(ii) The "more expensive than" ordering is induced by a linear price system.

In the language of this chapter:

(i) Agents have Ψ-convex preferences where Ψ is the set of all algebraic linear orderings (as shown in Section 3.3).
(ii) The "more expensive than" ordering $\geq_p$ on the set of alternatives (defined by $x \geq_p y$ if $p \cdot x \geq p \cdot y$) is a member of Ψ.

An important point is that the same set of primitive orderings appears in (i) and (ii) above. This suggests two new definitions. First, we enrich the notion of an economy with a set of primitives orderings Λ and require that all agents' preference relations are Λ-convex. We refer to such an economy by the term *convex economy*. Second, we refine the status equilibrium notion and require that the public ordering is one of the primitive orderings in Λ. We refer to such an equilibrium as a *primitive equilibrium*. Formally:

Definition: Convex Economy

A **convex economy** is a tuple $\langle N, X, (\succsim^i)_{i\in N}, F, \Lambda\rangle$ where $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ is an economy, Λ is a set of primitive orderings over X, and all preferences are Λ-convex.

Definition: Primitive Equilibrium

Let $\langle N, X, (\succsim^i)_{i\in N}, F, \Lambda\rangle$ be a convex economy. A **primitive equilibrium** is a status equilibrium $\langle \trianglerighteq, (x^i)_{i\in N}\rangle$ where $\trianglerighteq \in \Lambda$.

Obviously, any primitive equilibrium is a status equilibrium, and when Λ is the set of all orderings, any status equilibrium is a primitive equilibrium.

Example: The Give-and-Take Convex Economy

We return to the give-and-take economy. Recall that $X = [-1, 1]$ and F is the set of all profiles that sum up to 0. Let Λ consist of the two natural orderings: the rightist $\trianglerighteq_R$ (which favours taking) and the leftist $\trianglerighteq_L$ (which favours giving). Assume that every agent i holds continuous Λ-strictly convex preferences (with a single peak denoted by $peak^i$).

When $\Sigma_i peak^i = 0$, there is no conflict of interest in the economy and either primitive ordering, together with all agents choosing their peaks, is a primitive equilibrium. In fact, these are the only primitive equilibria (if $\langle \trianglerighteq_L, (x^i)_{i\in N}\rangle$ is an equilibrium, then $peak^i \le x^i$ for all i and therefore, $x^i = peak^i$ for all i).

A more interesting case is $\Sigma_i peak^i > 0$ where agents wish to take more than they wish to give. Let $F_{\leq}$ be the set of all feasible profiles with all agents at or to the left of their peaks.

We now verify that $F_{\leq}$ is equal to the set of all Pareto-optimal profiles. Any $(x^i) \in F_{\leq}$ is Pareto-optimal because any profile (y^i) that Pareto-dominates it must rank $x^i \leq y^i$ for all i, with at least one strict inequality; however, such a profile is infeasible because $0 = \Sigma x^i < \Sigma y^i$. On the other hand, if (x^i) is feasible and not in $F_{\leq}$, then $peak^i < x^i$ for some agent i and, by the assumption that $\Sigma_i peak^i > 0$, there is an agent j for whom $x^j < peak^j$. Transferring some small amount from i (who takes too much) to j (who gives too much) would be a feasible Pareto improvement.

The following claim shows that the First and Second Welfare Theorems hold for this example.

Claim: In the give-and-take convex economy with $\Sigma_i peak^i > 0$, the primitive equilibria are all tuples $\langle \trianglerighteq_R, (x^i)_{i \in N} \rangle$ where (x^i) is a Pareto-optimal profile.

Proof: A pair $\langle \trianglerighteq_L, (x^i)_{i \in N} \rangle$ cannot be a primitive equilibrium since then $peak^i \leq x^i$ for all i and thus $0 < \Sigma_i peak^i \leq \Sigma_i x^i$. If $\langle \trianglerighteq_R, (x^i)_{i \in N} \rangle$ is a primitive equilibrium, then (x^i) is in $F_{\leq}$, hence Pareto-optimal. Obviously, any pair $\langle \trianglerighteq_R, (x^i)_{i \in N} \rangle$ where (x^i) is in $F_{\leq}$ is a primitive equilibrium. ■

The rightist ordering reflects the norm that an agent should not take more (or not give less) than his assignment. This is a reasonable norm for governing a voluntary public fund in a society where the "aggregate" tendency of agents is to take rather than give. Under the alternative interpretation of status, the public is indoctrinated with the idea that giving less or taking more than expected is shameful.

Example: The Consensus Economy

In this example, different political positions are represented by points on a line. The (finite or infinite) set $X \subseteq \mathbb{R}$ consists of all possible political positions. As in the previous example, the set of primitive orderings Λ consists of the rightist and the leftist orderings, and every agent i has a continuous and Λ-strictly convex preference relation with a peak denoted by $peak^i$. Denote by l the leftmost peak and by r the rightmost peak, and assume that $-1 < l < r < 1$. Harmony requires consensus, and thus F is the set of all constant profiles. The set of Pareto-optimal profiles consists of all profiles $(x^*, \ldots, x^*)$ with $x^* \in [l, r]$.

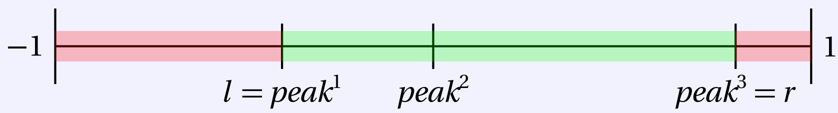

Figure 3.2 Primitive equilibrium positions – red; Pareto-optimal positions – green

The pair $\langle \trianglerighteq_L, (x^*, ..., x^*) \rangle$ is a primitive equilibrium if and only if $r \leq x^*$. Likewise, the pair $\langle \trianglerighteq_R, (x^*, ..., x^*) \rangle$ is a primitive equilibrium if and only if $x^* \leq l$. Therefore, except for the boundaries, the primitive equilibrium profiles and the Pareto-optimal profiles are almost completely disjoint. Thus, the First and Second Welfare Theorems both fail for this economy.

Example: A Convex Economy with No Primitive Equilibrium

Primitive equilibria can fail to exist even when a status equilibrium exists. Consider the convex housing economy with four agents, four houses arranged on a line $a - - b - - c - - d$, and $\Lambda = \{\trianglerighteq_L, \trianglerighteq_R\}$. Agents 1 and 2 hold the preferences $\trianglerighteq_L$, while 3 and 4 hold the preferences $\trianglerighteq_R$. A status equilibrium exists (for example, the profile (a, b, c, d) with the public ordering $aPdPbPc$). However, there are no primitive equilibria since any primitive ordering bottom-ranks an extreme alternative z

(which is either a or d) and since there are two agents who top-rank z, at least one of them is not assigned z and strictly prefers z, violating the definition of primitive equilibrium.

3.5 A First Welfare Theorem

We have just seen that the consensus economy has primitive equilibrium profiles that are not Pareto-optimal. The following are two other illuminating examples in which primitive equilibrium profiles can be non Pareto-optimal:

(i) In a single-agent convex economy, every feasible alternative x^* (preference-maximal or not) together with a critical direction of the preferences at x^* is a primitive equilibrium.
(ii) In a convex economy where all agents' preferences are equal to the same primitive ordering $\trianglerighteq$, every feasible profile (whether it is Pareto-optimal or not) combined with the public ordering $\trianglerighteq$ is a primitive equilibrium.

The following condition on *convex environments*, $\langle N, X, F, \Lambda \rangle$, plays a key role in explaining why the First Welfare Theorem is valid in the standard division economy but not in the above examples.

Definition: Condition D

A convex environment $\langle N, X, F, \Lambda \rangle$ satisfies **condition D** if there is no primitive ordering $\trianglerighteq$ and two distinct feasible profiles (a^i) and (b^i) such that $b^i \triangleright a^i$ or $b^i = a^i$ for all i.

Three prominent convex economies that satisfy condition D are: (i) the housing economy with any set of primitive orderings, (ii) the standard division economy with Λ being the set of all algebraic linear orderings and F requiring that all goods are fully allocated, and (iii) the give-and-take economy where Λ consists of the increasing and the decreasing orderings. The following proposition shows that condition D is necessary and sufficient for the First Welfare Theorem.

Proposition 3.2: A First Welfare Theorem

Let $\langle N, X, F, \Lambda\rangle$ be a convex environment.

(i) If the convex environment satisfies condition D, then for any profile of Λ-convex preferences $(\succsim^i)_{i\in N}$ any primitive equilibrium profile (a^i) of the convex economy $\langle N, X, (\succsim^i)_{i\in N}, F, \Lambda\rangle$ is weakly Pareto-optimal (there is no other feasible (b^i) such that for all i either $b^i \succ^i a^i$ or $b^i = a^i$).

(ii) If the convex environment fails condition D, then there are Λ-convex preferences $(\succsim^i)_{i\in N}$ such that the convex economy $\langle N, X, (\succsim^i)_{i\in N}, F, \Lambda\rangle$ has a primitive equilibrium profile that is not weakly Pareto-optimal.

Proof:

(i) Consider a primitive equilibrium $\langle \trianglerighteq, (a^i)\rangle$. If (a^i) is not weakly Pareto-optimal, then there is another feasible profile (b^i) such that for all i either $b^i \succ^i a^i$ or $b^i = a^i$. Then, for all i, either $b^i \triangleright a^i$ or $b^i = a^i$, contradicting condition D.

(ii) Since condition D fails, there exist two distinct feasible profiles, (a^i) and (b^i), and a primitive ordering $\trianglerighteq$ such that for all i, $a^i \triangleright b^i$ or $a^i = b^i$. Extend the convex environment to a convex economy by endowing each agent with the same Λ-convex preference relation $\trianglerighteq$. Then, $\langle \trianglerighteq, (b^i)\rangle$ is a primitive equilibrium that is not weakly Pareto-optimal.

Note that when condition D is satisfied, every primitive equilibrium profile is weakly Pareto-optimal, but it might be not Pareto optimal. For example, for the housing economy with two houses, two agents, preferences $a \succ^1 b$ and $a \sim^2 b$, and any set of primitive orderings, condition D holds, but $\langle a \triangleright b, (b, a)\rangle$ is a primitive equilibrium with a Pareto-nonoptimal profile.

3.6 A Second Welfare Theorem

We have seen that the Second Welfare Theorem does not generally hold. Essentially, it requires the following Richness property:

Definition: Richness

The convex economy $\langle N, X, (\succsim^i), F, \Lambda \rangle$ satisfies **Richness** if the following holds: Let (a^i) be a feasible profile and $\trianglerighteq^i$ and $\trianglerighteq^j$ be two different primitive orderings such that (recall the notation $B(\trianglerighteq, a^i) = \{x \mid a^i \trianglerighteq x\}$):
(i) a^i is $\succsim^i$-maximal in $B(\trianglerighteq^i, a^i)$ but not in $B(\trianglerighteq^j, a^i)$; and
(ii) a^j is $\succsim^j$-maximal in $B(\trianglerighteq^j, a^j)$ but not in $B(\trianglerighteq^i, a^j)$.
Then, there is a pair of alternatives $(b^i, b^j) \neq (a^i, a^j)$ such that:
(I) $(b^i, b^j, a^{-i,j}) \in F$ and (II) (b^i, b^j) Pareto-dominates (a^i, a^j)
(That is, $b^i \succsim^i a^i$ and $b^j \succsim^j a^j$ with at least one strict preference.)

The Richness property is illustrated in Figure 3.3 using an Edgeworth box. It states that, for any feasible profile (a^i), if $\trianglerighteq^1$ is a critical direction for agent 1 at a^1 and $\trianglerighteq^2$ is a critical direction for agent 2 at a^2, and the directions are not identical, then there is a feasible mutually beneficial reconfiguration of their bundles (b^1, b^2), which leaves all other agents unchanged.

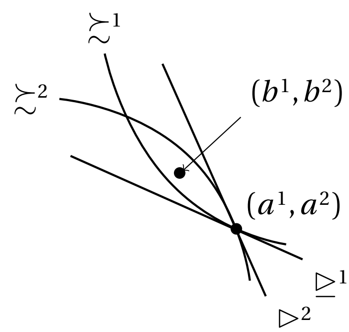

Figure 3.3 Richness

Proposition 3.3: A Second Welfare Theorem

Let $\langle N, X, (\succsim^i)_{i \in N}, F, \Lambda \rangle$ be a convex economy that satisfies Richness. Then, any Pareto-optimal profile is a primitive equilibrium profile.

Proof:

Let $(x^i)_{i \in N}$ be a Pareto-optimal profile. Let O^i be the set of all critical directions of $\succsim^i$ at x^i (that is, the set of all $\trianglerighteq \in \Lambda$ satisfying that for any z, if $z \succ^i x^i$, then $z \triangleright x^i$). By the Λ-convexity of the preferences, $O^i \neq \emptyset$.

If $\cap_i O^i$ were empty, then there would be two agents i and j such that O^i and O^j are non-nested sets. Take $\trianglerighteq^i \in O^i \setminus O^j$ and $\trianglerighteq^j \in O^j \setminus O^i$. The element x^i is $\succsim^i$-maximal in $B(\trianglerighteq^i, x^i)$ but not in $B(\trianglerighteq^j, x^i)$, and analogously for agent j. By the Richness property, there is a pair of elements (b^i, b^j) such that the modified profile obtained by replacing the pair (x^i, x^j) with (b^i, b^j) is feasible and Pareto-dominates $(x^i)_{i \in N}$, which contradicts the Pareto-optimality of (x^i). Thus, there exists $\trianglerighteq \in \cap_i O^i$, and therefore, $\langle \trianglerighteq, (x^i)_{i \in N} \rangle$ is a primitive equilibrium.

In Richter and Rubinstein (2015), it is shown that under the following two additional assumptions the Richness property is also necessary for the Second Welfare Theorem to hold: (i) differentiability of preferences and (ii) there are no two alternatives, x and x', such that $x \trianglerighteq x'$ for all primitive orderings $\trianglerighteq$.

3.7 Primitive Equilibrium – More Examples

Example: The Division Economy

Let $X = \mathbb{R}^L_+$ be the set of bundles in an L-commodity world. Let $z = (z_l)$ be the vector of total endowment which has to be fully divided, that is, (x^i) is feasible if $\Sigma_{i=1}^n x^i = z$. Let Ψ_+ be the set of all positive algebraic orderings, namely all $\geq_v$ where $v \in \mathbb{R}^L_+ \setminus \{0\}$. Let the set of primitive orderings Λ be a non-empty (finite or infinite) subset of Ψ_+. All agents hold monotonic Λ-convex preference relations. Two simple cases: When Λ contains a single ordering $\geq_v$, all agents hold the same preference relation, $\succsim = \geq_v$. When $\Lambda = \{\geq_{(1,0)}, \geq_{(0,1)}\}$, every indifference curve is "right-angled".

The First Welfare Theorem holds since the economy satisfies condition D and thus, by Proposition 3.2, any primitive equilibrium profile is weakly Pareto-optimal. The Richness property used in Proposition 3.3 holds and thus any Pareto-optimal allocation is a primitive equilibrium profile. This is somewhat stronger than the textbook Second Welfare Theorem which states that any Pareto-optimal allocation is an equilibrium allocation supported by *some* linear ordering while Proposition 3.3 states that the equilibrium public ordering can be drawn from Λ.

Example: The Collections Economy

A non-empty finite set of distinct indivisible goods Z is to be partitioned among the agents. Unlike in the housing economy, each agent chooses a *collection* of goods, which can have more than one good or none at all. Let X be the set of all subsets of Z. We will use lower-case letters for goods and the Greek symbols Θ and Φ for collections of goods. The set F contains all profiles that allocate each item in Z to exactly one agent. For every v, a positive-valued function on Z, let $\trianglerighteq_v$ be the ordering of X represented by the utility function $v(\Theta) = \Sigma_{z \in \Theta} v(z)$. That is, $v(\Theta)$ is the sum of the v-values attached to the individual items in the set Θ. Let Λ be the set of such strict orderings. As shown earlier, the Λ-convex preferences are exactly all preferences that are weakly monotonic with respect to the inclusion relation. We assume that all agents' preference relations are strict and Λ-convex.

In this economy, a primitive equilibrium has the interpretation that a price is attached to *each good* and the price of a collection is the sum of the prices of the goods in the collection. In contrast, a status equilibrium has the interpretation that there is a price for *each collection*.

Claim: For the collections economy: the set of primitive equilibrium profiles $\subseteq$ the set of Pareto-optimal profiles $\subseteq$ the set of status equilibrium profiles, and these inclusions can be strict.

Proof: To establish the first inclusion, by Proposition 3.2 it suffices to verify that condition D holds. Take a primitive ordering $\trianglerighteq_v$. For any two distinct feasible profiles, (Θ^i) and (Φ^i), it holds that $\Sigma_i v(\Theta^i) = \Sigma_i v(\Phi^i) = v(Z)$. Thus, it cannot be that $v(\Theta^i) \geq v(\Phi^i)$ for all i with at least one strict inequality.

However, there can be Pareto-optimal profiles that are not primitive equilibrium profiles. For example, let $Z = \{a, b, c, d\}$ and $n = 2$. Both agents have preferences that rank any cardinally larger set higher and are therefore, Λ-convex. To simplify notation, denote the set of goods $\{x, y\}$ as xy. Table 3.1 depicts the agents' preferences over two-element sets:

$\succsim^1$	$\succsim^2$
ac, bd	ad, bc
ab	cd
ad, bc, cd	ab, ac, bd

Table 3.1 Preferences with a Pareto-optimal profile that is not a primitive equilibrium profile (highlighted).

The profile $(x^1, x^2) = (ab, cd)$ is Pareto-optimal. However, there is no public ordering $\geq_v$ that supports this profile as a primitive equilibrium. If there were, then $ac >_v ab$ (to ensure that ab is optimal for agent 1), which implies that $v(c) > v(b)$. Similarly, we can conclude that $v(b) > v(d) > v(a) > v(c)$, a contradiction.

For the second inclusion, recall that by Proposition 3.1, any Pareto-optimal profile is a status equilibrium profile. However, there are collection economies with status equilibrium profiles that are not Pareto-optimal. For example, suppose that $Z = \{a, b, c, d\}$, $n = 2$, and both agents have identical Λ-convex preferences $\succsim^*$ satisfying that the sets ac and bd are $\succsim^*$-superior to ab and cd. Then, $\langle \succsim^*, (ab, cd) \rangle$ is a status equilibrium that is not Pareto-optimal. ■

This example demonstrates a stark contrast between equilibria with item-pricing (where the price of a bundle is the sum of the individual items' prices) and those with bundle-pricing (where a price is attached to each bundle). The following table summarizes the above claim:

	Item-pricing equilibria	Bundle-pricing equilibria
First Welfare Theorem	✓	X
Second Welfare Theorem	X	✓

Table 3.2 Depiction of the Claim

3.8 Initial Status Equilibrium

In this section, we extend the definition of a status equilibrium to cover extended economies. To remind the reader, an extended economy is an economy with the specification of an additional feasible profile $(e^i)_{i \in N}$ interpreted as an "initial profile". It specifies an alternative for each agent which he has the absolute right to choose, independently of other agents' choices and of the equilibrium parameters. When the alternatives are assets, the initial profile can be thought of as specifying initial ownership.

Definition: Initial Status Equilibrium

Given an extended economy $\langle N, X, (\succsim^i)_{i \in N}, F, (e^i)_{i \in N} \rangle$:
An **initial status equilibrium** is a pair $\langle P, (x^i)_{i \in N} \rangle$ where P is an ordering on X and (x^i) is a feasible profile such that every agent i's assigned alternative x^i is $\succsim^i$-optimal in his "budget set" $B(P, e^i) = \{x \in X \mid e^i P x\}$.

In an initial status equilibrium, an agent's choice set consists of all alternatives that are weakly P-inferior to his *initial alternative*. In contrast, in a status equilibrium, an agent's choice set consists of all alternatives that are weakly P-inferior to his *equilibrium alternative*.

Two comments:
(i) If x^i is $\succsim^i$-maximal in $B(P, e^i)$, then x^i is also $\succsim^i$-maximal in $B(P, x^i)$. Thus, any initial status equilibrium of an extended economy is also a status equilibrium of the underlying economy.
(ii) If $\langle P, (x^i) \rangle$ is a status equilibrium, then for every strict ordering P' which is a tiebreaking of P, the pair $\langle P', (x^i) \rangle$ is also a status equilibrium. This is not the case for an initial status equilibrium: Consider the extended housing economy with two houses a and b, two agents, initial profile $(e^1, e^2) = (a, b)$, and preference relations $b \succ^1 a$ and $a \succ^2 b$. The public ordering that equally ranks a and b and the profile (b, a) constitute an initial status equilibrium for the extended economy. However, breaking this indifference will invalidate the equilibrium since one agent will not be able to "afford" the other house.

Even though a status equilibrium exists when a Pareto-optimal profile does (Proposition 3.1), the following example demonstrates that the existence of an initial status equilibrium is not guaranteed even for finite extended economies.

Example: An Extended Jobs Economy

Consider the jobs economy of Section 3.2 with 3 agents, two jobs a and b, and capacities $n_a = 2$ and $n_b = 1$. Assume that agents 1 and 2 prefer b and agent 3 prefers a.

There are two Pareto-optimal profiles (b, a, a) and (a, b, a), both of which are status equilibrium profiles (with the public ordering bPa). If either of those profiles is the initial profile, then it is also an initial status equilibrium profile.

However, if the initial profile is (a, a, b), where each agent starts with the alternative he dislikes, then an initial status equilibrium does not exist. To see why, note that an equilibrium public ordering cannot rank a weakly above b, because then agents 1 and 2 would both choose b, violating feasibility. Nor can it be that b is ranked strictly above a, because then all three agents would choose a, again violating feasibility.

The "problem" is that the initial status equilibrium concept does not allow for the exchange of a and b between 1 and 3 (or between 2 and 3) due to the equilibrium concept's inability in allowing different budget sets for two agents with the same initial alternative.

The reader may wonder why no equilibrium exists in this extended economy whereas an equilibrium does exist in the standard competitive market model. The reason is that, in the standard competitive market model there is also money in the economy and a monetary amount can be attached to the transaction of exchanging a for b so that at least one of the two agents who prefer b to a would be indifferent between conducting the transaction or refraining from it. Then, the public ordering in the standard market is not merely ordinal but cardinal, indicating the monetary amount required to exchange a lower-ranked good for a higher-ranked one.

Thus, the existence of an initial status equilibrium is not guaranteed when the initial profile assigns identical elements to different agents. However, whenever every agent has a distinct initial alternative, the following proposition establishes the existence of an initial status equilibrium. Furthermore, it shows that if in addition there is a strict partial ordering R such that all individual preferences are R-monotonic (that is, if aRb then $a \succ^i b$ for all i), then there is an R-monotonic equilibrium public ordering. Taking R to be the empty binary relation gives the baseline result of Shapley and Scarf (1974) (presented in Section 1.3).

Proposition 3.4: Existence of an Initial Status Equilibrium

Any extended economy $\langle N, X, (\succsim^i)_{i\in N}, F, (e^i)_{i\in N}\rangle$ where all initial alternatives are distinct has an initial status equilibrium. If, in addition, all preference relations are R-monotonic with respect to a strict partial ordering R, then the public ordering can be taken to be R-monotonic as well.

Proof:

Let Y be the set of alternatives in the initial profile (e^i). For any $Z \subseteq Y$, define $M(Z) = \{i \mid e^i = z \text{ for some } z \in Z\}$ to be the set of agents initially assigned to alternatives in Z. Following the construction in Proposition 1.5, select a sequence of top trading cycles, $B^1,\ldots,B^T$. Define a partial ordering P on Y by aPb if $a \in B^t$, $b \in B^s$, and $t \le s$ (all elements in the same B^t are P-indifferent). We need to extend P to all of X. Partition $X \setminus Y$ into sets $A^1,\ldots,A^{T+1}$ as follows: For any $x \in X \setminus Y$, let $x \in A^t$ where t is the smallest index such that there is an agent $i \in M(B^t)$ who strictly prefers x over all elements in B^t. If there is no such t, then let $x \in A^{T+1}$. Place the elements in any A^t below B^{t-1} and above B^t. Define P on A^t as any arbitrary expansion of R. To see that P expands R everywhere, consider a and b such that aRb (and thus, all agents prefer a to b).

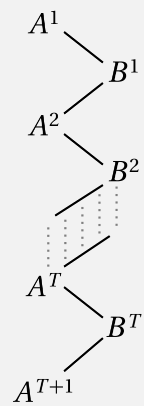

Figure 3.4 The Construction

- If $b \in Y$, then $b \in B^t$ for some t. If $a \in Y$, then it must belong to an earlier trading cycle because no agent would top-rank b when a is present, and thus, aPb. If $a \notin Y$, then the agent who top-ranks $b \in B^t$ prefers a to all elements of B^t. Therefore, $a \in A^s$ with $s \le t$, thus, aPb.
- If $b \in A^t$ for some $t \le T$, then for some $i \in M(B^t)$ it holds that $b \succ^i y$ for all $y \in B^t$. Thus, i also prefers a over all $y \in B^t$. If $a \in Y$, then a belongs to a previous trading cycle and if $a \notin Y$, then it belongs to A^s with $s \le t$. In either case aPb (for the case that $a, b \in A^t$, recall that P expands R on A^t).
- If $b \in A^{T+1}$ and $a \notin A^{T+1}$, then aPb and if $a \in A^{T+1}$, then aPb because P expands R on A^{T+1}.

The profile (y^i), which assigns to each agent $i \in M(B^t)$ the element y^i that e^i points to in the top trading cycle B^t, together with P, constitutes an initial status equilibrium: First, (y^i) is feasible since it is a permutation of the initial profile. Second, for every i, $e^i P y^i$ because y^i and e^i are in the same cycle. Third, suppose that $z \succ^i y^i$ for some agent i. Then, if $z \in Y$, it belongs to an earlier cycle. If $z \notin Y$, then i prefers z to all elements of B^t, and so $z \in A^s$ for $s \le t$. In either event, $z P y^i$.

Example: The Extended Give-and-Take Economy

Extend the give-and-take economy by adding an initial profile $(e^i)_{i \in N}$ that is feasible, $\Sigma e^i = 0$. Each agent i for whom $e^i > 0$ has the right to take e^i from the public fund, while each agent i for whom $e^i < 0$ has the right to contribute $-e^i$. Remember that every agent i has continuous and strictly convex (and thus single-peaked) preferences with a peak at $peak^i$. As before, we focus on the case where $\Sigma peak^i > 0$. Here, we adopt the interpretation that aPb means that b is more socially beneficial than a. Each agent chooses how much to give or take from the alternatives that are more socially beneficial than his initial assignment.

The existence of an initial status equilibrium for this extended economy is guaranteed by Proposition 3.4 only for the case that all e^i are distinct. Here, we construct a simple initial status equilibrium with an attractive structure which also demonstrates existence even when the e^i are not distinct.

Let P_z be the ordering that places all alternatives between -1 and z equally at the bottom and is strictly increasing from z to 1. Every agent i faces the interval budget set $[-1, \max\{z, e^i\}]$ and so has a unique optimal choice which is continuous in z, weakly increasing, and strictly increasing for $z \in [e^i, peak^i]$ (in the case that $e^i < peak^i$). Given the total indifference ordering P_1, every agent would choose $peak^i$, and the sum of their chosen actions would be $\Sigma_i peak^i > 0$. Given the strictly increasing ordering P_{-1}, every agent i chooses an alternative $x^i \le e^i$ and the sum of the chosen alternatives is non-positive since $\Sigma_i x^i \le \Sigma_i e^i = 0$. Thus, by the continuity of the agents' choices in z, there is a $z^* \in [-1, 1]$ for which the sum of the chosen elements is 0. The ordering P_{z^*} together with the profile of optimal choices from the corresponding budget sets is an initial status equilibrium.