NO PRICES NO GAMES!

FOUR ECONOMIC MODELS

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5 A Comparison to Game Theory

In this final chapter, we compare this book's modelling approaches to those of standard Game Theory. Rather than talking abstractly, we make the comparison more concrete by considering two specific "battlegrounds". In both, we contrast our solution concepts with the standard ones. As always, we *do not* argue positively or normatively in favour of adopting the social institutions underlying these solution concepts. Rather, we simply wish to encourage readers to abandon the dogmatic use of familiar solution concepts and to consider less conventional frameworks.

The first battleground is the matching economy. It involves a single even-numbered population of agents who must match into exclusive pairs, which we refer to as a *pairing*. Each agent is characterized by a preference relation over potential mates. A special case is Gale and Shapley (1962)'s two-sided matching problem, one of the most iconic models in Economic Theory. In that problem, the agents are partitioned into two equal-sized groups, and every agent prefers to be matched with any agent from the other group over being matched with an agent from his own group.

The standard cooperative game-theoretical solution concept for matching economies is "pairwise stability", which is a pairing for which there are no two agents in different pairs who prefer each other over their current partners. Following Richter and Rubinstein (2024), we compare this concept with three of the approaches discussed in this book (modified to fit the matching problem):

- The *jungle equilibrium* in which a power relation governs society.
- The *Y-equilibrium* in which society is governed by norms that specify what is permissible and what is forbidden.
- The *status equilibrium* and *initial status equilibrium* concepts in which a status ordering upholds harmony in a society.

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The second battleground is a "political economy" setting. There is a group of agents with views on a political issue. Each agent chooses a position and cares only about the position he himself chooses (and not about the outcome of the process). There is a need that a majority of agents choose the same position; otherwise, a crisis ensues.

Traditionally, such a situation is modelled as a non-cooperative game, and its *Nash equilibria* are calculated. Extending Richter and Rubinstein (2021), we compare this approach to two of the approaches discussed in this book:

- The *convex Y-equilibrium* in which society is governed by (convex) norms that specify what is permissible and what is forbidden.
- The *biased preferences equilibrium* in which where preferences are systematically biased.

On both battlegrounds, the matching problem and the political economy setting, we will see that the new approaches lead to very different outcomes than the traditional ones.

5.1 The Matching Economy

In the matching economy, N is an even-numbered population of n agents. The set of alternatives is taken to be the set of agents, i.e. $X = N$. A *pairing* is a profile $(x^i)_{i\in N}$ that specifies, for every i, a partner $x^i \neq i$, such that if i is paired with j, then j is paired with i. The feasibility set F is the set of all pairings. Note that F is not closed under permutations (if i and j exchange partners, then feasibility requires that x^i and x^j also do). A match between two agents i and j is denoted as $i \hookleftarrow\!\!\rightarrow j$. We assume that agents prefer to have any partner over being alone; in other words, every agent bottom-ranks himself. Therefore, it will be sufficient to specify, for each agent i, a strict preference relation $\succ^i$ over $X\backslash\{i\}$, the set of all *other* agents.

As mentioned, the standard solution concept for this economy is pairwise stability, which is a pairing such that there are no two agents in different pairs who prefer each other over their respective current partners. Formally, a pairing (x^i) is *pairwise stable* if there is no i and j such that $j \succ^i x^i$ and $i \succ^j x^j$.

The *two-sided matching economy* is a special case of the matching economy. The set of agents N is partitioned into two equally-sized groups, N_1 and N_2, and every agent prefers any agent from the other group over any agent from his own group. We say that a pairing is *mixed* if every couple has one member from each group. For two-sided matching economies, a pairwise-stable pairing always exists and can be calculated using Gale and Shapley (1962)'s deferred acceptance algorithm. However, in the general matching economy, a pairwise-stable pairing often does not exist. The following is Gale and Shapley (1962)'s canonical example of a matching economy with no pairwise-stable pairing:

Agent	1	2	3	4
1st Preference	2	3	1	1
2nd Preference	3	1	2	2
3rd Preference	4	4	4	3

Table 5.1 A matching economy with no pairwise-stable pairing.

To see that there is no pairwise-stable pairing, consider a candidate pairing. Let i be the agent matched with 4. Agent i prefers every other agent over 4, and there is an agent $j \in \{1,2,3\}\setminus\{i\}$ who top-ranks i. Thus, the couple $i \leftrightarrow j$ blocks the candidate pairing from being pairwise stable.

As discussed in Section 0.4, we distinguish between two types of equilibrium concepts: the choice type (such as competitive equilibrium) and the deviation type (such as Nash equilibrium). Two of the concepts which we will apply, the Y-equilibrium and the initial status equilibrium, belong to the choice type. In these solution concepts, some internally determined parameter will restrict every agent's choice set, such that there is a pairing in which every agent's partner is one he most prefers from his choice set.

The other solution concepts that we apply to the matching problem belong to the deviation type. An equilibrium concept of this type captures immunity to certain threats that would "rock the boat". Such a solution concept consists of a pairing and an internally determined parameter that restricts every agent's deviation possibilities, such that no agent can profitably deviate.

Pairwise stability belongs to the deviation class. Given a pairing, the threat to pairwise stability is a potential deviation by two agents who prefer each other over their current partners. We have something different in mind — unilateral threats: for harmony to be disturbed, it is sufficient that even one agent is willing and able to approach another. The threat is merely the *approach* of one agent to another who is not his partner, whether or not his approach is reciprocated.

The perspective of life which we model is that a pairing can be destabilized not by coalitions. It is sufficient that an agent A approaches an agent B (who is not matched with A) and expresses his desire that B abandons his current partner and matches with him instead. This destabilizes society regardless of whether or not B reciprocates A's affections. Why does A approach B? Actually, why not? He may know that B also prefers him over B's current partner (this is the premise of pairwise stability). Even if he knows that B does not prefer him, A might hope that if he approaches B, then B will feel flattered and change his mind. Finally, A might not know B's preferences and simply tries his luck.

Generally, it is impossible that every agent is paired with his top choice since agents' desires are not perfectly reciprocated. Therefore, achieving stability when agents can make unilateral approaches requires restrictions on which approaches are allowed. Given such restrictions, we say that a pairing is *unilaterally stable* if there is no agent who wishes to approach another and is able to do so. A familiar and very restrictive social norm forbids anyone from approaching any matched individual. Such a norm achieves harmony in a society, but at the cost of drastically curtailing personal freedom. The restrictions described in the following three sections involve social institutions (power, taboos, and status) that limit an agent's ability to act, but in a less draconian manner.

The following is a running example which will be used to illustrate the different solution concepts:

Example: The Common-ranking Two-sided Matching Economy

A *common-ranking two-sided matching economy* is a two-sided matching economy where every agent in N_1 ranks members of N_2 according to a common ranking $j_1 \succ_2 j_2 \succ_2 \ldots \succ_2 j_{n/2}$, and every agent in N_2 ranks his potential partners in N_1 according to $i_1 \succ_1 i_2 \succ_1 \ldots \succ_1 i_{n/2}$.

In such an economy, the set of Pareto-optimal pairings is the set of all mixed pairings (recall, a pairing is mixed if every agent is paired with an agent from the other side). To see this, any pairing that matches two members of the same side also matches two members on the other side. However, in that case, all four agents can be beneficially re-paired with members of the other side, which is a Pareto improvement. On the other hand, in any mixed matching, improving one agent's situation requires moving another agent down the common preference ladder; thus, no Pareto improvements exist.

Before proceeding, let us recall the classic serial dictatorship algorithm: There is an equal number of agents and objects and the agents are strictly ranked. The agents each choose an object according to their rank. That is, the highest-ranked agent selects his most-preferred object; then the second-highest agent selects his most-preferred object from those remaining, and so on until all agents have made a selection.

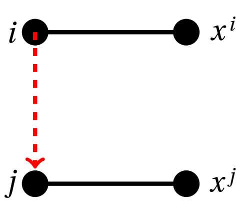
We modify this algorithm as follows: Again, the agents are strictly ordered, but this time, the highest-ranked agent selects a partner from among the other agents. Both he and his partner are removed, and the highest-ranked remaining agent selects a partner from among the remaining agents, and they are also removed. This algorithm proceeds until every agent either "chooses" or is "chosen".

Formally, a pairing (x^i) is the outcome of the *modified serial dictatorship procedure* with the ordering $\gg$ if there is a sequence of $n/2$ agents $i_1, \ldots, i_{n/2}$ such that i_1 is the $\gg$-maximal and x^{i_1} is his most-preferred agent from N, agent i_2 is the $\gg$-maximal and x^{i_1} is his most-preferred from $N - \{i_1, x^{i_1}\}$, and so on.

Denote by MSD, the set of pairings that result from the *modified serial dictatorship procedure* for some ordering $\gg$. Obviously, MSD is non-empty and any pairing in MSD is Pareto-optimal.

5.2 The Jungle Equilibrium

The solution concepts we apply here are related to the jungle model described in Chapter 1. In this section, an equilibrium candidate is a tuple $\langle \rhd, (x^i) \rangle$ where $\rhd$ is a strict ordering on N and (x^i) is a pairing. The statement $i \rhd j$ means that "i is more powerful than j". We consider several variants of the jungle equilibrium that differ in which circumstances power prevents an agent from approaching another. In a J1-equilibrium, an agent is only able to approach agents who are weaker than himself. In a J2-equilibrium, an agent needs to be stronger than both the agent he is approaching and that agent's current partner. In a J2*-equilibrium, he has to be stronger than the agent he is approaching and his own partner. In a J3-equilibrium, an agent needs to be stronger than all three of these agents: the agent he is approaching, his partner and his own partner. The following table summarizes the situations in which an agent i can approach agent j:



Concept	Power Requirement
J1	$i \rhd j$
J2	$i \rhd j$ and $i \rhd x^j$
J2*	$i \rhd j$ and $i \rhd x^i$
J3	$i \rhd j$ and $i \rhd x^j$ and $i \rhd x^i$

Figure 5.1 A potential approach by i to j (left panel) and the conditions under which such an approach can be made for each solution concept (right panel).

We now arrive to the formal definition of the J1-equilibrium concept.

Definition: J1-Equilibrium

A *J1-equilibrium* is a tuple $\langle \rhd, (x^i) \rangle$ in which there are no two agents i and j such that i prefers j over his current partner (that is, $j \succ^i x^i$) and i is more powerful than j (that is, $i \rhd j$).

In a J1-equilibrium pairing, it is possible that an agent A prefers another agent B to his current partner. But, agent A is prohibited from approaching B because B is stronger than him. In contrast, according to pairwise stability, what prevents agent A from approaching agent B is that B will reject him.

The following proposition states that the J1-equilibrium concept is stricter than both pairwise stability and MSD. Since pairwise-stable pairings do not always exist, neither will J1-equilibria.

Proposition 5.1: J1-equilibrium Properties

(i) Every J1-equilibrium pairing is both pairwise stable and an MSD outcome (and thus Pareto optimal).
(ii) A pairwise-stable pairing might not be a J1-equilibrium pairing.
(iii) An MSD pairing might not be a J1-equilibrium pairing.

Proof:

(i) Let $\langle \rhd, (x^i) \rangle$ be a J1-equilibrium. The profile (x^i) is clearly the outcome of the MSD procedure with the ordering $\rhd$. In order to show that (x^i) is pairwise stable, suppose that there are two agents i and j who strictly prefer each other to their current partners. One of them must be $\rhd$-stronger than the other, and he prefers the weaker agent over his current partner, thus violating the J1-equilibrium condition.

(ii) In a J1-equilibrium, the strongest agent is matched with his first-best choice. In the following matching economy, the (red) pairing 1↔2 and 3↔4 is pairwise stable, but no agent is matched with his first best:

Agent	1	2	3	4
1^{st} Preference	4	3	1	2
2^{nd} Preference	2	1	4	3
3^{rd} Preference	3	4	2	1

Table 5.2 Preferences with a pairwise-stable pairing (in red) and an MSD pairing (in blue), neither of which is a J1-equilibrium outcome.

(iii) In the economy presented in Table 5.2, the (blue) pairing $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$ is an MSD outcome (with the ordering $1 \gg 2 \gg 3 \gg 4$), but both 3 and 4 are matched with their worst partner, which cannot occur in a J1-equilibrium.

Example: The Common-ranking Two-sided Matching Economy

The pairing $\{i_1 \leftrightarrow j_1, i_2 \leftrightarrow j_2, \cdots, i_{n/2} \leftrightarrow j_{n/2}\}$ combined with any power ordering $\triangleright$ that satisfies $i_1, j_1 \triangleright i_2, j_2 \triangleright \cdots \triangleright i_{n/2}, j_{n/2}$ is a J1-equilibrium. There is no other J1-equilibrium pairing: Since every agent in N_2 top-ranks i_1, agent i_1 must be stronger than everyone in N_2 except perhaps his partner. This means that, in any equilibrium, i_1 has to be matched with his first-best, namely j_1. Similarly, j_1 must be stronger than all members of N_1, except possibly i_1. This pattern continues down the ranking. Among the remaining agents, i_2 and j_2 are matched, and i_2 must be more powerful than $\{j_3, \ldots, j_{n/2}\}$ while j_2 must be stronger than $\{i_3, \ldots, i_{n/2}\}$ and so on.

In a J1-equilibrium, the ability of one agent to approach another depends solely on the power relationship between them. However, in the context of the matching economy, any approach involves not only the agent who initiates the approach and the approached agent but also their partners. The following solution concepts take this into account.

Definition: J2-Equilibrium, J2*-equilibrium and J3-equilibrium

A *J2-equilibrium* is a tuple $\langle \triangleright, (x^i) \rangle$ for which there are no i and j such that $j \succ^i x^i$ and $i \triangleright j, x^j$.

A *J2*-equilibrium* is a tuple $\langle \triangleright, (x^i) \rangle$ for which there are no i and j such that $j \succ^i x^i$ and $i \triangleright j, x^i$.

A *J3-equilibrium* is a tuple $\langle \triangleright, (x^i) \rangle$ for which there are no i and j such that $j \succ^i x^i$ and $i \triangleright j, x^i, x^j$.

Obviously, every J1-equilibrium is also a J2-equilibrium as well as a J2*-equilibrium and every J2-equilibrium or J2*-equilibrium is a J3-equilibrium.

Part (i) of the following proposition proves that in terms of sets of outcomes, the three concepts are identical and equal to MSD. Thus, these equilibria always exist and their outcomes are Pareto optimal. Part (ii) shows that the concepts are neither weaker nor stronger than pairwise stability. Since pairwise-stable pairings are Pareto optimal, there can be a Pareto-optimal pairing which is not a J3-equilibrium outcome. Part (iii) shows that any ordering is a J3-equilibrium ordering (which is not the case for the J2- and J2*-equilibrium concepts).

Proposition 5.2: J2, J2* and J3-equilibrium Properties

(i) The sets of J2-equilibrium pairings, J2*-equilibrium pairings, and J3-equilibrium pairings are identical and equal to MSD (and thus such pairings always exist and are Pareto-optimal).

(ii) The set of pairwise stable outcomes and MSD do not include one another.

(iii) For any strict ordering $\gg$, there is a J3-equilibrium with the power relation $\gg$.

Proof:

(i) Choose an arbitrary strict ordering $\gg$ of the agents and apply the modified serial dictatorship procedure. Let $i_1, \ldots, i_{n/2}$ be the agents who make a choice according the procedure and $j_1, \ldots, j_{n/2}$ be the agents who are chosen where i_k chooses j_k for each k.

Any power relation satisfying $i_1 \rhd i_2 \rhd \cdots \rhd i_{n/2} \rhd j_1, \ldots, j_{n/2}$ supports this matching as a J2-equilibrium. Any agent who might be preferred by i_k over j_k must have been paired earlier, and thus, is either stronger than i_k or has a partner who is. No j_l can approach another agent because every other couple, $i_k \leftrightarrow j_k$, has at least one member who is stronger than him, namely i_k.

The power relation $i_1 \rhd^* j_1 \rhd^* i_2 \rhd^* j_2 \cdots \rhd^* i_{n/2} \rhd^* j_{n/2}$ supports the the matching as a J2*-equilibrium. No j_l can approach another agent because his partner i_l is stronger than he is. Any agent who might be preferred by i_k over j_k must have been paired earlier, and thus, is stronger than i_k.

To complete the proof it is sufficient to show that if $\langle \rhd, (x^i) \rangle$ is a J3-equilibrium, then (x^i) is the outcome of the MSD procedure with the ordering $\rhd$. Order the matches in this J3-equilibrium as follows: $a_1 \longleftrightarrow b_1, \ldots, a_{n/2} \longleftrightarrow b_{n/2}$ where $a_k \rhd b_k$ for all k and $a_1 \rhd a_2 \rhd \cdots \rhd a_K$. Thus, the pairing (x^i) will be the result of the MSD with the ordering $\rhd$.

(ii) For the matching economy depicted in Table 5.2, the red highlighted pairing is pairwise stable but is not in MSD since no agent gets his first-best. The blue highlighted pairing is in MSD but is not pairwise stable.

(iii) Let $\gg$ be a strict ordering and (x^i) be the MSD outcome with this ordering. Then, $\langle \gg, (x^i) \rangle$ is a J3-equilibrium. By the MSD procedure, half of the agents "make a choice" while the rest "are chosen". Any agent who "makes a choice" can only prefer agents who are matched before him, i.e. those who are stronger than him or are paired with a stronger partner. Any "chosen" agent is neutralized by being matched with a stronger partner.

Example: The Common-ranking Two-sided Matching Economy

Every mixed pairing (x^i) is a J2-equilibrium pairing (and thus also a J2*- and a J3-equilibrium outcome) supported by assigning the power relations of agents in each side by the rank of their partners (that is, for every two members i and j from the same side assign $i \rhd j$ if x^i is higher-ranked than x^j).

While every mixed pairing is part of a J2-equilibrium, not every power relation is. For example, in the case of four agents, there is no J2-equilibrium with the power relation $j_2 \rhd i_1 \rhd i_2 \rhd j_1$. This is because j_2 is the most powerful, and must be matched with i_1. Thus, the only candidate pairing is $\{i_1 \leftrightarrow j_2, i_2 \leftrightarrow j_1\}$. But this is not a J2-equilibrium because i_1 prefers j_1 over j_2, and is stronger than both i_2 and j_1.

5.3 Restricting Partnerships: Pairwise Y-equilibrium

We now adjust the Y-equilibrium concept (Chapter 2) to fit the matching economy. Since every agent needs a partner, uniformly restricting the set of permitted partners will leave some agents without a partner. Instead, we model social norms that determine which *pairs* are permitted and which are forbidden.

Definition: Y-Equilibrium

Let M be the set of all pairs. A *para-Y-equilibrium* is a tuple $\langle Y, (x^i) \rangle$ where $Y \subseteq M$ and (x^i) is a pairing such that, for every agent i, x^i is $\succ^i$-maximal in $\{j \mid i \leftrightarrow j \in Y\}$. A *Y-equilibrium* is a para-Y-equilibrium such that there is no other para-Y-equilibrium $\langle Z, (y^i) \rangle$ with $Y \subset Z$.

Any set of permissible pairs Y induces, for each agent i, a choice set of permissible partners $\{j \mid i \leftrightarrow j \in Y\}$. Thus, unlike in Chapter 2, here the Y-equilibrium notion treats agents asymmetrically in the sense that different agents face different choice sets with the restriction that if j is permissible for i, then i is permissible for j. The adapted Y-equilibrium notion requires that, for any larger permissible set, there is an i and j such that i would choose j but j would not choose i.

The following proposition shows that the set of Y-equilibrium pairings is the set of all Pareto-optimal pairings and thus, always exists.

Proposition 5.3: Y-equilibrium and Pareto Optimality

The set of Y-equilibrium pairings = The set of Pareto-optimal pairings.

Proof:

Given any pairing (x^i), define $L((x^i))$ to be the set of all pairs $i{\leftrightarrow}j$ such that $x^i \succsim^i j$ and $x^j \succsim^j i$. Notice that for every i, the pair $i{\leftrightarrow}x^i$ is in $L((x^i))$.

Let $\langle Y,(x^i)\rangle$ be a Y-equilibrium and (y^i) be a pairing that Pareto-dominates (x^i). Obviously, $L((x^i)) \supseteq Y$ and $L((y^i)) \supseteq L((x^i))$. In fact, the inclusion is strict since at least one agent, say j, is strictly better off in (y^i) and therefore the pair $j{\leftrightarrow}y^j$ is in $L((y^i)) - L((x^i))$. The tuple $\langle L((y^i)),(y^i)\rangle$ is a para-Y-equilibrium with a larger set of permissible pairs, which contradicts $\langle Y,(x^i)\rangle$ being a Y-equilibrium.

On the other hand, let (x^i) be a Pareto-optimal pairing. We now show that the tuple $\langle L((x^i)),(x^i)\rangle$ is a Y-equilibrium. If not, then there is a para-Y-equilibrium $\langle Z,(y^i)\rangle$ with $Z \supset L((x^i))$. All agents are weakly better off in (y^i) than in (x^i). The set Z contains at least one pair $i{\leftrightarrow}j$ which is not in $L((x^i))$. Without loss of generality, suppose that $j \succ^i x^i$. In that case, $y^i \succsim^i j \succ^i x^i$ and therefore (y^i) Pareto-dominates (x^i).

Note that this result does not rely on preferences being strict. The result implies that, when preferences are strict, in term of outcomes, the Y-equilibrium notion is more permissive than pairwise stability or the J-equilibrium notions. But, this relationship breaks down when preferences can have indifferences because pairwise-stable pairings need not be Pareto optimal.

Example: The Common-ranking Two-sided Matching Economy

Every mixed pairing is Pareto optimal and therefore is a Y-equilibrium pairing. The Y-equilibrium pairing $\{i_1{\leftrightarrow}j_1, i_2{\leftrightarrow}j_2, \ldots\}$ is supported by a maximally restricted permissible set which contains only the equilibrium matches (and all the matches between any two agents from the same side). On the other hand, the Y-equilibrium with pairing $\{i_1 \leftrightarrow j_{n/2}, i_2 \leftrightarrow j_{n/2-1}, \ldots\}$ is much more permissive since it permits not only the equilibrium matches, but also any match $i_k \leftrightarrow j_l$ if $k+l > n/2+1$.

5.4 Prestige by Partner: Status Equilibrium

We now turn to the status equilibrium concept discussed in Chapter 3 (referred to as an S-equilibrium in Richter and Rubinstein (2024)). Harmony is achieved by means of a status ordering of the agents that blocks an agent from approaching certain other agents. In a status equilibrium, agents are paired up, and *no agent can approach any other with higher status than his current partner*. The only agents he has the courage to approach are those with a (weakly) lower status than his own partner. An equilibrium is harmonious in that no agent can find a different partner who is both more desirable and less prestigious than his partner.

In line with the alternative interpretation of public orderings (Chapter 3), a status ordering could represent "anti-prestige" whereby *lower-ordered agents are more prestigious*. Under this interpretation, a prestige-focused agent is only willing to consider leaving his current partner for more prestigious partners (i.e. those who are *lower* by the status ranking than his current partner).

Definition: Status Equilibrium

A status equilibrium is a tuple $\langle P, (x^i) \rangle$ where (x^i) is a pairing and P is a weak ordering of the agents such that, for every agent i, there is no j such that $j \succ^i x^i$ and $x^i P j$.

One more interpretation of a status equilibrium is a kind of "double ownership". For any pair $A \leftrightarrow B$, both A owns B and B owns A. The status ordering represents a notion of value whereby each agent "owns" his partner, and can "exchange" him for any weakly "less expensive" agent. In a status equilibrium, no agent wishes to do so.

Note that, a key difference between the roommate problem and the economic settings of Chapter 3 is that there the set of feasible profiles was closed under all permutations which does not hold for the roommate problem. Thus, the results from that chapter are not applicable to the roommate problem considered here.

One property that does hold for the roommate problem is that: At least one agent (one with the highest-ranked partner) is matched with his most-preferred partner, and if preferences are strict, then at most one agent gets his least-preferred partner (this can only happen to the agent with the lowest-ranked partner).

We now establish some relationships between the status equilibrium concept and the other solution concepts.

Proposition 5.4: Status Equilibrium Properties

(i) Every status equilibrium pairing is an MSD outcome (and therefore a J2-equilibrium outcome and Pareto-optimal) but the reverse does not necessarily hold.
(ii) The notions of status equilibrium and pairwise stability are distinct; it is possible for either to exist when the other does not.

Proof:

(i) Let $\langle P, (x^i) \rangle$ be a status equilibrium and break ties so that P is strict. Define an ordering $\gg$ on the agents by the status of their partners: $i \gg j$ if $x^i P x^j$. The matching (x^i) is the outcome of the MSD procedure with the relation $\gg$, and thus is a J2-equilibrium outcome and Pareto optimal.

In the economy depicted in Table 5.2, the blue highlighted pairing is in MSD, but it is not a status equilibrium pairing because agents have strict preferences and two agents are matched with their least-preferred partners.

(ii) In the economy depicted in Table 5.1, there is no pairwise-stable pairing, but the ordering $1\,P\,2\,P\,3\,P\,4$ supports the pairings $\{1{\leftrightarrow}3, 2{\leftrightarrow}4\}$ and $\{1{\leftrightarrow}4, 2{\leftrightarrow}3\}$ as status equilibria.

In the economy depicted in Table 5.2, the red highlighted pairing $\{1{\leftrightarrow}2, 3{\leftrightarrow}4\}$ is pairwise stable, but there is no status equilibrium: The highlighted pairing is not a status equilibrium pairing because no agent gets his first-best. Neither are the other two pairings because in each of them, two agents are matched with their last choice.

Example: The Common-ranking Two-sided Matching Economy

By Proposition 5.5, only mixed pairings can be status equilibrium pairings. In fact, *any mixed pairing* is a status equilibrium pairing with any ranking P that satisfies $i_1 P i_2 P \ldots P i_{n/2}$ and $j_1 P j_2 P \ldots P j_{n/2}$ (for any agent i, every agent that i desires more than his partner has a higher status than i's partner).

5.5 Prestige by Self: Initial Status Equilibrium

We adapt the initial status equilibrium concept to the matching economy by taking an agent's initial status to be himself. This concept belongs to the choice group of solution concepts (see Section 0.4). Every agent chooses his partner, but the status ranking only allows an agent to approach agents with the same status or lower. In equilibrium, the status ranking is such that the individual choices form a pairing (that is, if i chooses j, then j chooses i). This adapted initial status equilibrium concept is referred to as a C-equilibrium in Richter and Rubinstein (2024).

Definition: Initial Status Equilibrium

An initial status equilibrium is a tuple $\langle P, (x^i)\rangle$ where P is a status ordering of the agents and (x^i) is a pairing such that for every agent i, his partner x^i is i's most-preferred partner in $\{j \in N \mid iPj\}$.

Obviously, any two matched agents in an initial status equilibrium must have the same status. Therefore, every initial status equilibrium is also a status equilibrium and, by Proposition 5.5, its pairing is Pareto optimal.

Of particular interest is the relationship between the initial status equilibrium and the J1-equilibrium. If $\langle P, (x^i)\rangle$ is an initial status equilibrium, then $\langle \rhd, (x^i)\rangle$ is a J1-equilibrium where $\rhd$ is any strict tie-breaking of P (that is, $i \rhd j$ only if iPj).

However, unlike the initial status equilibrium concept, a J1-equilibrium does not require that an agent be weakly stronger than his partner. Thus, a J1-equilibrium pairing need not be an initial status equilibrium pairing. For example, consider the two-sided matching economy with $N_1 = \{1, 3\}$, $N_2 = \{2, 4\}$, and a tragedy: 1 loves 2, 2 loves 3, 3 loves 4, and 4 loves 1. Both mixed pairings are J1-equilibrium pairings. One is the first-best for N_1's members and is supported by any power relation that ranks N_1's members above N_2's members. The other is the opposite. Neither is an initial status equilibrium pairing (shortly, we will see why).

We need one more concept. We say that the matching economy is *pair-rankable* if the set of agents N can be partitioned into doubletons $I_1, \ldots, I_{n/2}$, such that for every q, each agent in I_q prefers his partner in the doubleton to any member of $I_{q+1} \cup \cdots \cup I_{n/2}$. In other words, the agents can be partitioned into a sequence of doubletons in which every agent's partner is his best choice from those who are not ahead of him.

Pair-rankability is a strong property of a matching economy which emerges in some natural settings (see Alcalde (1994)). Two classical families of pair-rankable matching economies are:

(i) Agents live in a metric space and rank partners by their distance (closer is better). The first doubleton can consist of the two closest agents, and each subsequent doubleton consists of the two closest among those remaining.
(ii) Agents are positioned on a line, and each has single-peaked preferences over the other agents with a peak at one of his neighbours. This implies that an extreme agent top-ranks his only neighbour and, for any set of agents, there are two neighbours such that the left one top-ranks his right neighbour while the right one top-ranks his left neighbour.

Proposition 5.5: Initial Status Equilibrium Properties

(i) A matching economy has an initial status equilibrium if and only if it is pair-rankable.
(ii) If an initial status equilibrium pairing exists, then it is unique and a J1-equilibrium outcome (and hence pairwise stable).

Proof:

(i) Let $\langle P, (x^i) \rangle$ be an initial status equilibrium. An agent who is P-maximal must be matched with someone who is also P-maximal. Thus, both must like each other more than they like anyone else. Denote this doubleton as I_1. Among the agents outside of I_1, there is an agent who is P-maximal. Again, he and his partner are equally ranked and therefore, both must prefer each other to anyone else in $N - I_1$, and so on. Thus, the existence of an initial status equilibrium requires that the economy is pair-rankable.

In the other direction, let $I_1, \ldots, I_{n/2}$ be a partition of N into doubletons such that both agents in I_q prefers the other to any member of $I_{q+1} \cup \cdots \cup I_{n/2}$. To construct an initial status equilibrium, match each agent with his doubleton's partner and define a ranking P by iPj if i belongs to a weakly lower-indexed doubleton than j.

(ii) Given a pair-rankable matching economy with a partition $I_1, \ldots, I_{n/2}$ of N, the pairing in which every agent is matched with his doubleton's partner is the only initial status equilibrium pairing. This is because the two agents in I_1 must be matched in any initial status equilibrium since they top-rank each other and, for any equilibrium ranking, one of them can "afford" the other. The same argument then applies to the agents in I_2 and so on. Obviously, this pairing with the ranking P from part (i) is a J1-equilibrium.

Example: The Common-ranking Two-sided Matching Economy

The economy is uniquely pair-rankable with $I_q = \{i_q, j_q\}$. By Proposition 5.6, the unique initial status equilibrium is the pairing $\{i_1 \leftrightarrow j_1, i_2 \leftrightarrow j_2, \ldots\}$ with the status ranking P that ranks xPy if $x \in I_p$, $y \in I_q$ and $p \leq q$.

5.6 Comparing the Approaches

Figure 5.2 summarizes the relationship between the different equilibrium concepts, pairwise stability (PS), Pareto optimality (Pareto) and the modified serial dictatorship procedure outcomes (MSD). The lines in the diagram stand for inclusions (each inclusion can be strict). In particular, all equilibrium pairings are Pareto optimal (the First Welfare Theorem); only the Y-equilibrium satisfies the Second Welfare Theorem; and only the J2-, J3-, and Y-equilibria are guaranteed to exist. The symbols S and IS stand for status equilibrium and initial status equilibrium, respectively.

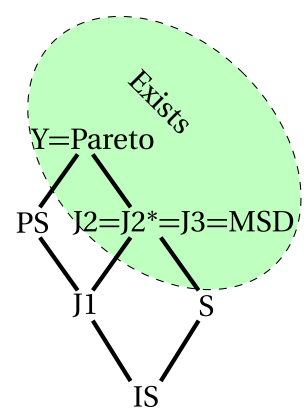


Figure 5.2 Relationship between the concepts.

One can imagine solution concepts other than those analysed in this chapter. An example is due to Herings and Zhou (2024) (see also Herings (2024)). They model a social restriction on the approaches that an agent can make. They require that for any pair of agents, the norm allows at least one of them to approach the other. In an equilibrium, every agent approaches his optimal partner from among those he is allowed to approach.

Definition: EE-equilibrium

An *EE-equilibrium* is a pairing $(x^i)_{i\in N}$ and a profile of permissible sets of agents $(A^i)_{i\in N}$ such that:

(i) For every two agents i and j, either $i \in A^j$ or $j \in A^i$ (or both).

(ii) For every i, agent x^i is i's most-preferred partner from A^i.

It is shown now that the EE-equilibrium outcomes are identical to the pairwise-stable pairings.

Proposition 5.6: EE-equilibrium = Pairwise Stable

The set of EE outcomes is equal to the set of pairwise-stable pairings.

Proof

Let (x^i) be a pairwise-stable pairing. Define $A^i = \{z : x^i \succsim^i z\}$. By definition, for every i, x^i is $\succsim^i$-maximal within A^i. In addition, for any two unmatched agents i and j, by pairwise stability it must be that $x^i \succsim^i j$ or $x^j \succsim^j i$, and therefore $j \in A^i$ or $i \in A^j$. Thus, $\langle (A^i), (x^i) \rangle$ is an EE-equilibrium.

Let $\langle (A^i), (x^i) \rangle$ be an EE-equilibrium. Then, for every i and j who are not matched, it must be that $i \in A^j$ and so $x^j \succsim^j i$, or $j \in A^i$ and so $x^i \succsim^i j$. Therefore, (x^i) is pairwise stable.

5.7 The Majority Voting Economy

We now turn to compare our approach to that of Non-Cooperative Game Theory. As mentioned, our battleground is the following *majority voting* economy: There is an odd number of agents ($n \geq 3$), each of whom chooses a position in $X = [-1,1]$. Each agent i has continuous and strictly convex preferences over X with a unique peak, denoted by $peak^i$. All peaks are distinct and we can assume that $-1 < peak^1 < peak^2 < \cdots < peak^n < 1$. To simplify notation, denote the leftmost peak, the median peak, and the rightmost peak as L, M, and R, respectively. The set $F \subseteq X^N$ consists of all profiles for which at least $\tau = (n+1)/2$ members choose the same position. If a profile (x^i) has a point shared by at least τ members, we denote it by $O((x^i))$ and refer to it as the *overall position* (clearly, it is not possible to have two such positions).

The situation we have in mind is one where societal harmony requires that a majority of members declare the same position and if there is no such majority, then a crisis bursts. An example is a committee or a jury who, according to their procedure, must come out with a position that is supported by a majority of members. Another example is of a political party whose leaders must take positions on an issue. To prevent the public from becoming confused and abandoning the party, at least a majority of them need to take the same position.

Note that in Hotelling (1929) and its many extensions, an agent cares only about the group's position, while in Downs (1957), an agent also cares about his chosen position. We go a step further and assume that an agent cares only about his chosen position and does not care at all about the group's majority position.

5.8 Convex Y-equilibrium

First, we analyze the economy's *convex Y-equilibria* (Chapter 2). Recall that a convex Y-equilibrium is a configuration $\langle Y, (y^i) \rangle$ (where Y is a convex subset of X and (y^i) is a profile of choices from Y) that satisfies the three conditions:

(i) Rationality: for all i, y^i is a $\succsim^i$-maximal position in Y.

(ii) Feasibility: $(y^i) \in F$.

(iii) Set maximality: there is no convex set $Z \supset Y$ and profile $(z^i) \in F$ such that z^i is a $\succsim^i$-maximal alternative in Z for all i.

Recall that, in any convex Y-equilibrium, the permissible set is closed (if not, then the closure of the permissible set with the same profile is a larger convex para-equilibrium). Since Y is convex and closed and all preference relations are strictly convex, every agent's maximal position is unique.

We now show that there are exactly two convex Y-equilibria in this economy and both have the overall position M (the median peak): a "rightist" equilibrium in which M and all positions to its right are permissible and a "leftist" equilibrium in which M and all positions to its left are permissible (see Figure 5.2).

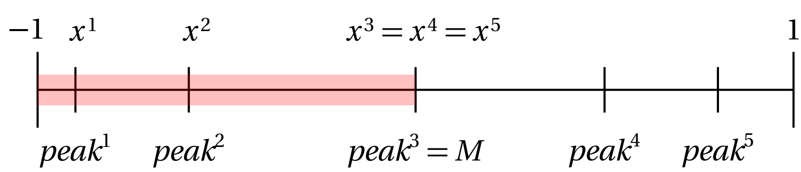


Figure 5.3 A leftist equilibrium

Proposition 5.7: Convex Y-equilibria in the Majority Voting Economy

In the majority voting economy, there are two convex Y-equilibria. Their permissible sets are $[-1, M]$ and $[M, 1]$. Both have the overall position M.

Proof:

The set $[-1, M]$ is a convex para-equilibrium permissible set because all of the rightist agents and the median voter vote M, thereby constituting a majority. Likewise, $[M, 1]$ is a convex para-equilibrium permissible set.

To show that $[-1, M]$ and $[M, 1]$ are convex Y-equilibrium permissible sets and that there are no others, it suffices to show that any convex para-equilibrium permissible set is a subset of either $[-1, M]$ or $[M, 1]$. To see this, note that if a convex permissible set contains points to both the left and right of M, then no position attracts majority support: M must be in the permissible set since it is convex and the median agent selects it, all leftist agents (a minority) choose positions to the left of M and all rightist agents (also a minority) choose positions to the right of M. Thus, every convex Y-para-equilibrium's permissible set is contained in $[-1, M]$ or $[M, 1]$.

At first glance, Proposition 5.7 appears to be a kind of "median voter theorem" since the only convex Y-equilibrium overall position is the median. Thus, in terms of outcomes, the convex Y-equilibrium involves a compromise; however, the price of this "happy ending" is that only positions to one side of M are permitted.

5.9 Biased Preferences Equilibrium

To fit the model into Chapter 4's definition of an economy, we modify the specification of the agents' preferences as we did for the give-and-take economy. Assume that each agent i has two considerations in mind, labelled l and r, and maximizes the utility function $u^i(x) = u^i_l(x) + u^i_r(x)$ where u^i_l is strictly decreasing and represents an argument for leftist positions, while u^i_r is strictly increasing and represents an argument for rightist ones. The functions u^i_l and u^i_r are differentiable with non-zero and finite derivatives at each point. They are also strictly concave, which implies that their sum induces convex preferences with a unique peak, denoted by $peak^i$, and we assume that all peaks are different.

A bias (λ_l, λ_r) transforms an agent i's utility function into $\lambda_l u_l^i(x)+\lambda_r u_r^i(x)$. Consequently, a leftist bias (where $\lambda_l > \lambda_r$) moves the peak of every agent to the left, while a rightist bias (where $\lambda_l < \lambda_r$) moves all of the peaks to the right. There always exist extreme biased preferences equilibria where a majority of agents agree on the rightmost position supported by a rightist bias, which increases the weight of the rightist consideration strongly enough that at least a majority of individuals become extreme rightists. Likewise, there are also extreme leftist biased preferences equilibria.

It is possible that there are biased preferences equilibria where a majority of individuals move in one direction, and it just so happens that a majority of the biased peaks coincide but this would be a fluke occurrence. However, there is never a biased preferences equilibrium with a majority at the median position (since the peaks are distinct, a bias is necessary for agreement, but if there is a rightist bias, then any equilibrium must have a right-of-median overall position, and vice-versa if there is a leftist bias).

To summarize, extreme positions are always biased preferences equilibrium overall positions and the median never is, whereas the unique Y-equilibrium overall position is the median.

5.10 The Majority Voting Game and Nash Equilibrium

To model the majority voting economy $\langle N, X, (\succsim^i)_{i\in N}, F\rangle$ as a *strategic game*, let the set of players be N, and let each player's set of actions be the set of positions X. Each player j has a preference relation $\succsim_*^j$ on the set of all *choice profiles*, defined by $x=(x^i) \succsim_*^j y=(y^i)$ if either:

(i) $x \in F$ and $y \notin F$; or

(ii) both $x, y \in F$ or both $x, y \notin F$ and $x^j \succsim^j y^j$.

In other words, every agent's lexicographical first priority is harmony, and his second priority is his own position.

We apply the standard Nash equilibrium to this game. We distinguish between non-crisis Nash equilibria (in which a majority of players agree on a position) and crisis Nash equilibria (in which no overall position exists). Recall that n is odd. If $n > 3$, then there is a unique crisis equilibrium in which every agent chooses his own peak. No other crisis equilibria exist since, if the outcome of the game is a crisis, then every agent chooses his peak, because otherwise any agent could profitably deviate to his peak, whether or not that results in harmony. If $n = 3$, then a crisis equilibrium does not exist since any agent can deviate to one of the other two positions and thus, avoid a crisis.

In this game, the notion of a non-crisis Nash equilibrium is identical to that of the *social equilibrium* in Debreu (1952)'s model of generalized games (see Tóbiás (2022) for a review of conditions guaranteeing its existence). A social equilibrium is a profile of actions in F such that every player's action is a best response *from among the set of actions that are available to him given the other players' actions*. In other words, the profile after the deviation must be in F. Formally, $(x^i) \in F$ is a *social equilibrium* if for each i, the action x^i is optimal for i from among all the actions t^i such that $(t^i, x^{-i}) \in F$.

The difference between the Nash and Debreu formulations is purely semantic: every player is either not interested in moving from a non-crisis profile to a crisis profile (in the Nash formulation) or is not even allowed to do so (in Debreu's formulation).

All profiles in which a bare majority of exactly $\tau = (n+1)/2$ agents choose the same position (whatever it is), while the rest choose their peaks, are non-crisis Nash equilibria. These equilibria can be extremely unnatural in that the coalition which supports them does not have anything to do with the position being supported. In particular, there are non-crisis Nash equilibria for *any* overall position, even extreme ones that are outside of $[L, R]$, and the agents supporting the overall position need not be those whose peaks are closest to it. We will now see that there are no other non-crisis Nash equilibria.

Proposition 5.8: Nash Equilibrium in the Voting Game

If $n \geq 5$, then the set of non-crisis Nash equilibria in the voting game consists of all profiles for which there is a position chosen by exactly τ agents while the rest choose their peaks.

Proof:

These are Nash equilibria: No agent at the majority position can deviate profitably since, if he did so, then a crisis would ensue because his former position would no longer be a majority position and neither would his new position (all other agents are choosing their peaks which are distinct, so any new position would have at most two agents, but $n \geq 5$). All other agents are at their first-best, they choose their peak, and no crisis occurs. Therefore, they do not want to deviate.

To see that there are no other non-crisis Nash equilibria, consider a Nash equilibrium in which at least τ agents choose a common position t. An agent who does not choose t is not critical in maintaining harmony and therefore, must be at his peak. If strictly more than τ agents choose t, then at least one of them is not at his peak and could deviate profitably.

Comment: In Richter and Rubinstein (2021), we conducted similar comparisons and reached similar conclusions regarding other conditions for "holding the group together":
(i) a consensus among a super majority of agents,
(ii) all positions are sufficiently close to the median position, or
(iii) all positions are sufficiently close to the average position.

5.11 Comparing our Approaches with Nash Equilibrium

The above analysis clarifies the significant differences between the convex Y-equilibrium, the biased preferences equilibrium, and the Nash equilibrium of the above political game. For the convex Y-equilibrium concept, M is the only overall position. For the biased preferences equilibrium concept, typically only the extreme positions, -1 and 1, are overall positions. In contrast, for the Nash equilibrium concept, all positions, even those outside the range $[L, R]$, are overall positions. Furthermore, a convex Y-equilibrium is "monotonic" in the sense that if agent i's ideal position is to the left of j's then his chosen position is weakly to the left of j's. In contrast, there are *always* non-monotonic Nash equilibria. The biased preferences equilibrium case is less clear: the existence of a non-monotonic equilibrium depends on the underlying utility functions.

Notice that the Nash equilibria require a high degree of coordination between the agents. In contrast, the Y-equilibrium and biased preferences equilibrium concepts only require that agents know either the social restrictions or biases, but not the behaviour of others. This is like the marketplace where individuals only need to know prices, but not other agents' actions.

Let us emphasise: we are not saying that the standard game-theoretical approach is "wrong", nor do we insist that the Y-equilibrium or biased preferences equilibrium approaches are "right". Rather, and as already mentioned, we are suggesting that the reader not automatically apply Nash-equilibrium-like concepts but instead considers alternative solution concepts in the spirit of those described in this book.